Lecture Notes on Mathematical Economics

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This version: Summer 2018

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1 Notations

1.1 Sets

You can either express a set by characterizing it with a property or by simply listing all elements of it.

$$
\begin{aligned}
A &= \{x : x \text{ satisfies property } \boldsymbol{A}\} \\
A &= \{x_1, x_2, \ldots\}
\end{aligned}
$$

Example 1.1 (Russell's Paradox)**.** Let $A = \{x : x \notin x\}$. Then $A \in A \Leftrightarrow A \notin A$.

For solutions that fix this paradox, please refer to axiomatic set theory. Let X be a restricted universe given which such paradoxes do not arise and consider only sets whose elements are in X.

Example 1.2. Conventional notations:

$$
\begin{aligned}
\mathbb{N} &= \{1, 2, 3, \ldots\}, \text{ the set of natural numbers.} \\
\mathbb{Z} &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}, \text{ the set of integers.} \\
\mathbb{Q} &= \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}, \text{ the set of rational numbers.} \\
& \quad \mathbb{R}, \text{ the set of real numbers.}
\end{aligned}
$$

Definition 1.1. A set A is a subset of X if $x \in A \Rightarrow x \in X$, written as $A \subset X$.

Definition 1.2. For any subsets A and B of X, we define

1. $A \cap B$, the intersection of A and B, by $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$,
2. $A \cup B$, the union of A and B by $A \cup B = \{x \in X : x \in A \text{ or } x \in B\}$,
3. $A \subset B$, A is a subset of B, if $x \in A \Rightarrow x \in B$,
4. $A = B$, A is equal to B, if $A \subset B$ and $B \subset A$,
5. $A \backslash B$, the difference between A and B, by $A \backslash B = \{x \in A : x \notin B\}$,
6. $A \Delta B$, the symmetric difference between A and B, by $A \Delta B = (A \backslash B) \cup (B \backslash A)$,

7. A^c, the complement of A, by $A^c = \{x \in X : x \notin A\}$,

8. $\emptyset$, the empty set, by $\emptyset = X^c$,

9. A and B to be disjoint if $A \cap B = \emptyset$.

All these relationships between sets can be easily depicted in Venn Diagrams.

Theorem 1.1. *For sets A, B, C,*

1. $A \cap B = B \cap A, A \cup B = B \cup A$;

2. $(A \cap B) \cap C = A \cap (B \cap C), (A \cup B) \cup C = A \cup (B \cup C)$; *and*

3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Theorem 1.2 (DeMorgan's Laws)**.** *If A, B are any sets, then $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$.*

Definition 1.3. For any subset A of X, the power set of A, denoted by $P(A)$ or 2^A, is the set of all subsets of A.

1.2 Products and relations

Note that the set $\{a, b\} = \{b, a\}$. We say that $\{a, b\}$ is an unordered pair. An ordered pair is one that distinguishes the first and second elements in a pair. So if (a, b) and (c, d) are ordered pairs, then $(a, b) = (c, d)$ iff $a = c$ and $b = d$. One way to define (a, b) is to let

$$(a, b) = \{\{a\}, \{a, b\}\}.$$

Definition 1.4. For any sets A and B, the Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs

$$\{(a, b) : a \in A, b \in B\}.$$

Example 1.3. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ defines the 2-dimensional Euclidean space, i.e., the plane. Similarly, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$ defines the n-dimensional Euclidean space.

A point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a vector, where x_i is a real number for each $i = 1, \ldots, n$. And x_i is called the i-th coordinate of the vector x.

Definition 1.5. A binary relation R between A and B is a subset $R \subset A \times B$, such that aRb iff $(a, b) \in R$.

Example 1.4. Let $A = \{0, 1, 2\}$. The relation $<$ between A and A can be defined by the subset $\{(0, 1), (0, 2), (1, 2)\} \subset A \times A$.

Example 1.5. The graph of any function (or mapping) $f : A \to B$, denoted by $\{(a, f(a) : a \in A)\}$, is a subset of $A \times B$. Therefore, each function between A and B is a relation between A and B. Also, for any subset R of $A \times B$, if for each $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in R$, then R is a function between A and B.

Definition 1.6. An equivalence relation on a set A (i.e., between A and A) is a relation $\sim$ that is

1. reflexive: $\forall a \in A, a \sim a$,
2. symmetric: $\forall a, b \in A, a \sim b \Leftrightarrow b \sim a$,
3. transitive: for all $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

Example 1.6. Equality is an equivalence relation on $\mathbb{R}$.

If $u : X \to \mathbb{R}$ is a utility function representing preferences on X, then defining $x \sim y$ by $u(x) = u(y)$ gives the indifference equivalence relation.

Definition 1.7. A relation R on X is complete if for all $x, y \in X, xRy$ or yRx, it is transitive if for all $x, y, z \in X, xRy$ and yRz implies xRz, it is rational if it is both complete and transitive.

Example 1.7. Let $X = \mathbb{R}$. Then $\leq$ is complete and transitive on $\mathbb{R}, <$ and $=$ are transitive but not complete, and $\neq$ is neither transitive nor complete.

Let $X = \mathbb{R}^2$ and define $\leq$ by $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. Then $\leq$ is not complete. In this case, we call $\leq$ a partial order on $\mathbb{R}^2$.

Example 1.8. From consumption theory, suppose $\succsim$ is a rational preference on a finite set X, then there exists a utility function $u : X \to \mathbb{R}$ that represents $\succsim$.

2 Differentiation

2.1 Euclidean space

The k dimensional Euclidean space is a vector space over the real field.

Definition 2.1. For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, x_2, ..., x_k),$$

where $x_1, ..., x_k$ are real numbers, called the coordinates of $\mathbf{x}$. Each $\mathbf{x} \in \mathbb{R}^k$ is called a vector. If $\mathbf{y} = (y_1, .., y_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$, let

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, ..., x_k + y_k), \\ \alpha \mathbf{x} &= (\alpha x_1, ..., \alpha x_k). \end{aligned}$$

Recall that the definition of a vector space involves only the closedness of addition and scalar (from the real field) multiplication.

The inner product of $\mathbf{x}$ and $\mathbf{y}$ on $\mathbb{R}^k$ is defined by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i,$$

and it induces the Euclidean norm

$$||\mathbf{x}|| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{k} x_i^2}.$$

Note that $|| \cdot ||$ is the standard notation for norm.

2.2 Derivatives

Definition 2.2. Let f be a real function on $[a, b]$. For any $x \in [a, b]$ if the limit

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists, define it to be $f'(x)$, and $f'(x)$ is called the derivative of $f(x)$ at x.

Higher-order derivatives can be defined similarly. And intuitively, $f'(x)$ gives the rate of change of $f(x)$ per unit change in x, and it equals to the slope of the tangent line of $f(x)$ at x. The second derivative, $f''(x)$ gives the rate at which the slope of f is changing, so it describes the curvature of the function f. Denote the n-th order derivative of f as $f^{(n)}$.

Definition 2.3. If $f^{(n)}$ exists and is continuous on $[a, b]$, we say that f is n-th order continuously differentiable, and denote it as $f \in C^n([a, b])$.

Theorem 2.1. *If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Proof. As $t \to x$, we have

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x}(t - x) \to f'(x) \cdot 0 = 0.$$

□

Example 2.1. A function that is continuous at but not differentiable at some point. Consider $f(x) = |x|$. It is not differentiable at 0.

Now we state some rules of differentiation without giving proofs. Please refer to Rudin (1976, Chapter 5) for details.

Theorem 2.2. *Suppose f and g are defined on $[a, b]$ and are differentiable at $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and*

(a) $(f + g)'(x) = f'(x) + g'(x)$;

(b) $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$;

(c) $(f/g)(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$.

If f is differentiable on the range of g and is differentiable at $g(x)$, then

(d) Chain rule $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$.

2.3 Mean value theorems

Definition 2.4. Let f be a real function defined on a metric space X. We say that f is a local maximum (minimum) at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq (\geq) f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

The following theorem gives what is called the first-order condition in economics.

Theorem 2.3. *Let f be defined on $[a, b]$; if f has a local maximum (minimum) at a point $x \in (a, b)$, and if f' exists, then $f'(x) = 0$.*

Proof. Choose δ such that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Hence as the left-hand limit, $f'(x) \geq 0$. And similarly, as the right-hand limit $f'(x) \leq 0$. Proved. □

The mean value theorem:

Theorem 2.4. *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof. Consider the function

$$d(x) = f(x) - [(\frac{f(b) - f(a)}{b - a})(x - a) + f(a)],$$

which measures the distance between $f(x)$ and the line through $(a, f(a))$ and $(b, f(b))$. Since $d(a) = d(b) = 0, d(x)$ must have a local maximum or minimum in (a, b). Let it be x. Then from the theorem above, $d'(x) = 0$. This completes the proof. $\square$

We know that if f is continuous on $[a, b]$, and $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$. Intuitively, this property comes from the continuity of the function f. However, for derivatives on an interval, this property always holds.

Theorem 2.5. *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < c < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = c$.*

Proof. Let $g(t) = f(t) - ct$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a, b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g has a local minimum achieved at some point $x \in (a, b)$, and it must be that $g'(x) = 0$. Therefore, $f'(x) = c$. $\square$

2.4 Functions with several variables

2.4.1 General differentiation

Let S be an open subset of $\mathbb{R}^n$.

Definition 2.5. A function $f : S \to \mathbb{R}^m$ is said to be differentiable at a point $x \in S$ if there exists a $m \times n$ matrix A such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{A}\mathbf{h}|}{|\mathbf{h}|} = 0,$$

and we write

$$D\mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x}) = \mathbf{A}.$$

2.4.2 Partial derivatives and directional derivatives

Again let S be an open subset of $\mathbb{R}^n$.

Definition 2.6. Let $f : S \to \mathbb{R}$. Then the partial derivative of f with respect to x_i is defined as

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots x_i, \ldots x_n)}{h},$$

given that the limit exists.

Sometimes the notation of $\frac{\partial f(\mathbf{x})}{\partial x_i}$ is shortened to be $f_i(\mathbf{x})$. Intuitively, $\frac{\partial f(\mathbf{x})}{\partial x_i}$ gives the rate of change of f when $\mathbf{x}$ is changing one unit in the direction of the i-th coordinate.

If we would like to know how f would change if we move away from $\mathbf{x}$ by a little towards the direction of some vector $\mathbf{h}$, we need the concept of directional derivative.

Definition 2.7. Let $f : S \to \mathbb{R}$. Let $\mathbf{x}$ be any point in S, and $\mathbf{h}$ any point in $\mathbb{R}^n$. Then the directional derivative of f at $\mathbf{x}$ in the direction $\mathbf{h}$ is

$$Df(\mathbf{x}; \mathbf{h}) \equiv \lim_{t \to 0+} \frac{f(\mathbf{x} + t\mathbf{h}) - f(\mathbf{x})}{t}.$$

given that the limit exists.

Theorem 2.6. *Suppose f is differentiable at x, then $Df(\mathbf{x}; \mathbf{h}) = Df(\mathbf{x}) \cdot \mathbf{h}$.*

Note that $Df(\mathbf{x}; \mathbf{h}) = \frac{df(\mathbf{x}+t\mathbf{h})}{dt}|_{t=0}$. This theorem can be directly proved by applying the chain-rule of multi-variable differentiation, a simple version of which states that

$$\frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \sum_{i=1}^{n} x_i'(t) \frac{\partial}{\partial x_i} f(x_1(t), \ldots, x_n(t)).$$

For $f : S \to \mathbb{R}$, the vector $Df(\mathbf{x}) = (\partial f / \partial x_1, \ldots, \partial f / \partial x_n)$ is also called the gradient of f at $\mathbf{x}$, denoted as $\nabla f(\mathbf{x})$.

2.4.3 Higher-order derivatives

If $f : S \to \mathbb{R}$ is differentiable on all of S, then $Df = (\partial f / \partial x_1, \ldots, \partial f / \partial x_n)$ defines a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. If the partial derivative of $\partial f / \partial x_i$ with respect to x_j exists at $\mathbf{x}$, denote it as $\partial^2 f / \partial x_i \partial x_j$ or f_{ij}. If f_{ij} exists for every pair of i, j, then we say that f is twice-differentiable at $\mathbf{x}$, and write

$$D^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix}.$$

If f is twice-continuously differentiable at $\mathbf{x}$, then $D^2 f(\mathbf{x})$ is also called the Hessian matrix, and denoted as $H(\mathbf{x})$. The following theorem states that Hessian matrix is symmetric.

Theorem 2.7. *If $f : S \to \mathbb{R}$ is a C^2 function, then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{x}),$$

for all $1 \leq i, j \leq n$, and $\mathbf{x} \in S$.

2.5 Implicit function theorem

A function of the form $y = y(x)$ is called an explicit function. In comparison, the equation $f(x, y) = c$, where c is a constant, defines y as an implicit function of x. Suppose two variables x and y satisfy the implicit relationship $f(x, y) = c$. The general question is under which conditions can we express y as an explicit function of x (or vice versa).

For an important application, consider the indifference curve $u(x, y) = c$. If this relation implies a function $y = y(x)$, then we see straightforwardly how should y (the consumption of the second commodity) change with x (the consumption of the first commodity) to maintain the same utility level c. Moreover, if we know that for some differentiable function $y(x), u(x, y(x)) = c$, then by taking derivative w.r.t. x for both sides of this equation, we have $u_x(x, y(x)) \cdot 1 + u_y(x, y(x)) \cdot y'(x) = 0$. (Note that we don't need to solve for $y(x)$ before doing this.) That is, at any x_0,

$$y'(x_0) = -\frac{u_x(x_0, y_0)}{u_y(x_0, y_0)},$$

where $u_x(x_0, y_0) \equiv (\partial u(x, y)/\partial x)(x_0, y_0)$. The economic interpretation is that to maintain the same utility level, the marginal rate of substitution (MRS) is equal to the ratio between the marginal utilities of the two commodities.

Example 2.2. Consider $f(x, y) = x^2 + y^2 - 1$. The equation $f(x, y) = 0$ specifies the unit circle on the plane. At any point (x_0, y_0) other than $(-1, 0)$ and $(1, 0)$, there is an open neighborhood of (x_0, y_0) on which the unit circle is part of a function $y = y(x)$. Note that at $(-1, 0)$ and $(1, 0)$, $f_y(x, y) = 0$.

Theorem 2.8 (Implicit function theorem, two-variable case). *Let (x_0, y_0) be a point on the locus of $f(x, y) = c$ in the plane, where f is a C^1 (continuously differentiable) function of two variables. If $f_y(x_0, y_0) \neq 0$, then $f(x, y) = c$ defines a C^1 function $y = y(x)$ in some neighborhood of (x_0, y_0).*

To see the intuition, note that for y to be a function of x, whenever y changes, x must also change (not the converse). Since $f(x, y) = c$, as long as the change in y adequately changes $f(x, y)$–as ensured by $f_y(x, y) \neq 0$–then x has to change accordingly to compensate for the change of y so that the value of f remains c. For a simple counterexample, consider the function $f(x, y) = c$ for all y.

Proof. We provide only a sketch of the proof. Assume $c = 0$. We know that $f_y(x_0, y_0) \neq 0$; assume it is positive. Since $f_y(x, y)$ is continuous on $\mathbb{R}^2$, there is a small neighborhood $N_\varepsilon((x_0, y_0))$ of (x_0, y_0) such that for any $(x, y) \in N_\varepsilon((x_0, y_0))$, $f_y(x, y) > 0$. Now, first, since $f(x_0, y_0) = 0$, and f is strictly increasing in y, there must be small number δ such that $f(x_0, y_0 + \delta) > 0$ and $f(x_0, y_0 - \delta) < 0$. Second, since f is continuous around $(x_0, y_0 + \delta)$, $f(x, y_0 + \delta)$ is positive for x close enough to x_0, and $f(x, y_0 - \delta)$ is negative for x close enough to x_0. Given each such x, there must be a unique $y = y(x)$ between $y_0 - \delta$ and $y_0 + \delta$ such that $f(x, y) = 0$. This is because f is continuous and strictly increasing on y. The rest of the work is to show that $y(x)$ is continuously differentiable. $\square$

Example 2.3 (Inverse function). Consider a C^1 function $y = g(x)$. Let $f(x, y) = y - g(x)$. Then $f(x, y) = 0$ iff $y = g(x)$, and if $f(x(y), y) = 0$ for some function $x(y)$, then $x(y)$ must be the inverse function of g, i.e., $x(y) = g^{-1}(y)$.

The implicit function theorem thus states that the inverse function of g exists around (x_0, y_0) on the locus of g if $f_x(x_0, y_0) \neq 0$, i.e., if $g'(x_0) \neq 0$. And $(g^{-1})'(y_0) = 1/g'(x_0)$.

To see the intuition, suppose $g'(x_0) > 0$. Then since g is continuously differentiable, there exists $\varepsilon > 0$ such that $g'(x) > 0$ for all $x \in N_\varepsilon(x_0)$. This implies that g is strictly increasing on $N_\varepsilon(x_0)$ and therefore is invertible on it.

For multi-variable functions, consider $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$. Then $f(x, y) = c$ describes the following equation system

$$\begin{cases} f_1(x_1, \ldots, x_m, y_1, \ldots, y_n) = c_1 \\ \qquad\qquad\vdots \\ f_n(x_1, \ldots x_m, y_1, \ldots, y_n) = c_n. \end{cases}$$

Theorem 2.9 (Implicit function theorem)**.** *Let (x_0, y_0) be a point on the locus of $f(x, y) = c$, where $f : A \times B \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function. Let $D_y f(x, y)$ be the $n \times n$ matrix whose ij-th element is*

$$\frac{\partial f_i(x, y)}{\partial y_j}, i, j = 1, \ldots, n.$$

Then if $D_y f(x_0, y_0)$ is invertible, there exist open sets U and V with $x_0 \in U \subset A$ and $y_0 \in V \subset B$, and a C^1 onto function $y : U \to V$ such that

$$f(x, y(x)) = c, \forall x \in U.$$

Once we know that $f(x, y) = c$ defines a C^1 function $y(x)$ around (x_0, y_0), we can then derive $y'(x_0)$, which is a $n \times m$ matrix, via the chain-rule: $y'(x)|_{x_0} = -[D_y f(x, y)]^{-1} \cdot D_x f(x, y)|_{(x_0, y_0)}$.

The intuition behind this theorem is similar to that of the two-variable case. For $y = (y_1, \ldots, y_n)$ (or equivalently, each $y_j, j = 1, \ldots n$) to be a function of x, whenever y_j changes, x must also change. Note that for each $j = 1, \ldots, n$, the marginal value of y_j on f is

$$\frac{\partial f(x, y)}{\partial y_j} = (\frac{\partial f_1(x, y)}{\partial y_j}, \ldots, \frac{\partial f_n(x, y)}{\partial y_j}),$$

which is the j-th column of the matrix $D_y f(x, y)$. Since $D_y f(x_0, y_0)$ is invertible, the vector $\partial f(x_0, y_0)/\partial y_j$ cannot be written as a linear combination of the other columns $\partial f(x_0, y_0)/\partial y_k, k \neq j$. That is, changes in y_k's, $k \neq j$, cannot compensate for the change of y_j to make the value vector of f remain c. Consequently, when y_j changes, x has to change accordingly.

Example 2.4. Suppose $f(x, y) = x + Ay$, where $x, y \in \mathbb{R}^2$ and A is a 2×2 matrix. Then

$f(x, y) = c$ defines a linear equation system

$$\begin{aligned} f_1(x, y) &= x_1 + a_{11}y_1 + a_{12}y_2 = c_1 \\ f_2(x, y) &= x_2 + a_{21}y_1 + a_{22}y_2 = c_2. \end{aligned}$$

We can express y as a function of x if A is invertible. (Due to linearity, $f_y(x, y) = A$ for all y.) In fact, if so,

$$y = A^{-1}(c - x).$$

3 Basic convex analysis

3.1 Convex sets

Both convex sets and functions have general importance in economic theory, not only in optimization.

Given two points $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$, the weighted average $\alpha x + (1-\alpha)y$ is called the convex combination of x and y. (As a simplification, in this lecture we use normal fonts instead of bold to denote vectors.)

Definition 3.1. A subset $S \subset \mathbb{R}^n$ is convex if for any $0 \leq \alpha \leq 1$ and $x, x' \in S$, $\alpha x + (1-\alpha)x' \in S$.

It's straightforward to see that if S is convex, then any finite convex combination $\alpha_1 x_1 + \cdots + \alpha_k x_k$ of points in S such that $\sum_{l=1}^{k} \alpha_l = 1$ also belongs to S, and the intersection of any number of convex sets is convex.

Definition 3.2. The convex hull of a set $S \subset \mathbb{R}^n$, denoted by $Conv(S)$, is the smallest convex set containing S.

Geometrically, $Conv(S)$ consists of all points that can be written as finite convex combinations of points in S. That is, it is the convex span of S.

Definition 3.3. For a set $S, x \in S$ is an extreme point of S if x cannot be written as the convex combination of other points in S.

Theorem 3.1 (Krein-Milman)**.** *A compact and convex set S is the convex hull of its extreme points.*

3.2 Hyperplane separation theorem

Definition 3.4. Let $p \in \mathbb{R}^n$ and $p \neq 0$, and let $a \in \mathbb{R}$. The set

$$H(p,a) = \{x \in \mathbb{R}^n : p \cdot x = a\}$$

is called a hyperplane in $\mathbb{R}^n$. The sets $\{x \in \mathbb{R}^n : p \cdot x \geq a\}$ and $\{x \in \mathbb{R}^n : p \cdot x \leq a\}$ are called the half-space above and below the hyperplane $H(p,a)$, respectively.

Recall that $p \cdot x = ||p|| \cdot ||x|| \cdot \cos\theta$, where θ is the angle between the vectors p and x. The vector p is the normal of the hyperplane $H(p, a)$ and is thus orthogonal (perpendicular) to it, and a half-space is above $H(p, a)$ if it is by the side of the hyperplane as the direction of p points to. Each hyperplane in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ with dimension $n-1$. A hyperplane in $\mathbb{R}^2$ is a straight line and a hyperplane in $\mathbb{R}^3$ is just a two-dimensional plane.

Example 3.1. If $p = (p_1, p_2) \in \mathbb{R}^2$ and $I \in \mathbb{R}$, then the hyperplane $H(p, I) = \{(x_1, x_2) \in \mathbb{R}^2 : p_1x_1 + p_2x_2 = I\}$ can be understood as the budget line, given income I and prices p_1, p_2.

Definition 3.5. Let X, Y be subsets of $\mathbb{R}^n$. We say that the hyperplance $H(p, a)$ separates X from Y if

$$p \cdot x < a, \forall x \in X \text{ and } p \cdot y > a, \forall y \in Y.$$

Theorem 3.2 (Hyperplane separation)**.** *Let C be a closed and convex subset of $\mathbb{R}^n$, and $x \in \mathbb{R}^n \backslash C$. Then there exists $p \in \mathbb{R}^n, p \neq 0$, and $a \in \mathbb{R}$ such that the hyperplance $H(p, a)$ separates C and x.*

Proof. Let y be the point in C that the distance between x and y, $||x - y||$, obtains its minimum. (Such y exists due to the Weierstrass theorem). If we let

$$p = x - y \text{ and } a' = p \cdot y,$$

then $p \cdot (x - y) > 0$, and hence $p \cdot x > a'$.

Next, we show that for all $z \in C, p \cdot z \leq p \cdot y$. Suppose this is not true. Consider $z \in C$ such that $p \cdot z > p \cdot y$. For any $\alpha \in (0, 1)$, let

$$w_\alpha = \alpha z + (1 - \alpha)y.$$

As C is convex, w_α is always in C. When α is sufficiently close to zero,

$$\begin{aligned}
||x - w_\alpha||^2 &= ||x - y + \alpha(y - z)||^2 \\
&= ||p + \alpha(y - z)|| \\
&= ||p||^2 + 2p \cdot \alpha(y - z) + \alpha^2||y - z||^2 \\
&= ||p||^2 + \alpha[2p \cdot (y - z) + \alpha||y - z||^2] \\
&< ||p||^2 = ||x - y||^2.
\end{aligned}$$

The last inequality is due to $p \cdot (y - z) < 0$ and that it is independent of α.

Therefore,

$$||x - w_\alpha|| < ||x - y||,$$

which contradicts with y being an point in C that is the closest to x. Therefore, $p \cdot z \leq a'$ for all $z \in C$.

Lastly, we can see that there must be $\varepsilon > 0, \varepsilon$ small enough such that $p \cdot z < a' + \varepsilon, \forall z \in C$ and $p \cdot x > a' + \varepsilon$. Let $a = a' + \varepsilon$, then $H(p, a)$ strictly separates C and x. □

There are many other versions of hyperplane separation theorems on how hyperplanes separate points or (open or closed) convex subsets from (open or closed) convex subsets; these theorems all share the same geometric intuition as that of the theorem above. When one of the two sets is not closed, the separation theorem may ensure only weak separation.

Example 3.2. Let $u_1(x)$ and $u_2(x)$ be utility functions of two agents defined on a convex set X. Suppose $\{v \in \mathbb{R}^2 : (v_1, v_2) \leq (u_1(x), u_2(x)) \text{ for some } x \in X\}$ is convex (e.g., this holds true when u_1, u_2 are concave functions, which we will formally define very soon). The following result is on the Pareto frontier of allocations between two agents; it can be generalized to more agents.

Claim 3.3. *Every Pareto efficient allocation $x^* \in X$ maximizes $\alpha u_1(x) + (1 - \alpha) u_2(x)$ for some $\alpha \in [0, 1]$.*

We now sketch the proof of this claim using a version of the Hyperplane separation theorem. Suppose x^* is Pareto efficient. Then the sets $A = \{v : v \leq u(x) \text{ for some } x \in X\}$ and $B = \{v : v > u(x^*)\}$ are disjoint. Since A is convex by assumption and B is also convex, there exist a vector $p \in \mathbb{R}^2$ and a scalar a such that $H(p, a)$ separates the two sets: that is, $p \cdot v \leq a$ for all $v \in A$ and $p \cdot v \geq a$ for all $v \in B$ (since B is not closed, we don't get strict separation).

Since the half-space above $H(p, a)$ contains B, it must be that $p \geq 0$. Geometrically, since $u(x^*) \in A$ and is on the boundary of B, we have $u(x^*) \in H(p, a)$. That is, $p \cdot u(x^*) = a \geq p \cdot v, \forall v \in A$. Hence $p \cdot u(x^*) \geq p \cdot u(x), \forall x \in X$. Let $\alpha = p_1/(p_1 + p_2)$; then we have a proof.

The hyperplane separation theorems are also applied in proving for example, the second theorem of welfare economics, and in game theory, in showing that a mixed strategy is never a best-response if and only if it is strictly dominated. In general, the theorem is potentially useful when we need to show the existence of a vector of weights.

Definition 3.6. The support function $h_X : \mathbb{R}^n \to \mathbb{R}$ of a closed and convex subset $X \subset \mathbb{R}^n$ is defined as

$$h_X(p) = \sup\{p \cdot x : x \in X\},$$

where $h_\emptyset(p) \equiv -\infty$ and $h_X(p) \equiv +\infty$ if the sup is infinite.

The support function $h_X(\cdot)$ is uniquely determined by X, and by definition, $X \subset \{x \in \mathbb{R}^n : p \cdot x \leq h_X(p)\}$. That is, X is included in the half-space below the hyperplane $H(p, h_X(p))$. Furthermore, due to the notation sup in the definition, $h_X(p)$ is chosen to be the smallest number a such that the half-space below $H(p, a)$ includes X.

Geometrically, for any closed and convex set X and a vector $p, h_X(p)$ is chosen in a way that the hyperplane $H(p, h_X(p))$ is tangent to X and X is included in the half-space below the hyperplane. That is, in direction $p, H(p, h_X(p))$ supports X from the above.

Claim 3.4. $h_X(p) = \inf\{a \in \mathbb{R} : p \cdot x \leq a, \forall x \in X\}$.

Example 3.3. Consider $X \subset \mathbb{R}^2$. If $X = \{x\}$, then $h_X(p) = p \cdot x$, and if $X = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$, then $h_X(p) = ||p||$.

Equivalently, we can define $h_X(p) = \inf_{x \in X} p \cdot x$. In this case, $H(p, h_X(p))$ supports the set X from the below.

Example 3.4. The expenditure function $e(p, u) = \inf_{x \in X_u} p \cdot x$ is the support function of the upper contour set $X_u = \{x : u(x) \geq u\}$ that corresponds to the utility level u.

Theorem 3.5 (Support-function theorem)**.** *For a closed and convex subset $X \subset \mathbb{R}^n$,*

$$X = \cap_{p \in \mathbb{R}^n} \{x \in \mathbb{R}^n : p \cdot x \leq h_X(p)\}.$$

Since this result is geometrically straightforward, we don't give its proof. Intuitively, it says that if we know that $h_X : \mathbb{R}^n \to \mathbb{R}$ is the support function of some closed and convex subset X of $\mathbb{R}^n$, then we can uniquely recover X by taking intersection of the half-spaces below the respective hyperplane $H(p, h_X(p)), p \in \mathbb{R}^n$. And in practice, we don't need to consider all $p \in \mathbb{R}^n$ when taking the intersection: it is sufficient to consider all directions p in the unit ball $\{x \in \mathbb{R}^n : ||x|| = 1\}$.

3.3 Concave functions

Definition 3.7. Let $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is convex. The function f is

(a) concave if $f(\alpha x + (1-\alpha)x') \geq \alpha f(x) + (1-\alpha)f(x')$, for any $x, x' \in S$ and $\alpha \in (0,1)$, and is

(b) convex if $f(\alpha x + (1-\alpha)x') \leq \alpha f(x) + (1-\alpha)f(x')$, for any $x, x' \in S$ and $\alpha \in (0,1)$

That is, a function is concave if its value at the average of two points is always great than or equal to the average of its values at the two points. When the inequality is strict for all x, x', α, then f is said to be strictly concave (or respectively, strictly convex). The functions which are both concave and convex are of the form $f(x) = a \cdot x + b$ and are called affine functions.

Example 3.5. The function $f : \mathbb{R}_+ \to \mathbb{R}$ defined as $f(x) = x^a$ is strictly concave if $0 < a < 1$ and is strictly convex if $a > 1$. The function $f : \mathbb{R}^2_+ \to \mathbb{R}$ defined as $f(x_1, x_2) = x_1^a x_2^b$ is called the Cobb-Douglas function; it is concave if $a + b \leq 1$ and is neither concave nor convex if $a + b > 1$.

Example 3.6. The expenditure function $e(p, u) = \inf_{x \in X_u} p \cdot x$ is concave in p.

For any concave function $f, -f$ is convex and vice versa. Hence it is without loss of generality to concentrate on concave functions. The result below can be viewed as an alternative way of defining concave functions.

Theorem 3.6. *Let $A \equiv \{(x, y) : x \in S, f(x) \geq y\}$ be the set of points "on and below" the graph of f. Then f is concave if and only if A is a convex set.*

Proof. ($\Rightarrow$) Suppose f is concave. Let $(x_1, y_1), (x_2, y_2)$ be two points in A. Then by definition, $f(x_1) \geq y_1$ and $f(x_2) \geq y_2$. Since f is concave, for any $\alpha \in [0, 1]$,

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &\geq \alpha f(x_1) + (1-\alpha)f(x_2) \\ &\geq \alpha y_1 + (1-\alpha)y_2. \end{aligned}$$

This means that $(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2)$, which is $\alpha(x_1, y_1) + (1-\alpha)(x_2, y_2)$, also belongs to A. Hence A is a convex set.

($\Leftarrow$) Suppose A is a convex set. Let x_1, x_2 be two points in S. Since $f(x_1) \geq f(x_1)$ and $f(x_2) \geq f(x_2)$, we have

$$(x_1, f(x_1)), (x_2, f(x_2)) \in A.$$

As A is convex, for any $\alpha \in [0, 1]$,

$$\alpha(x_1, f(x_1)) + (1 - \alpha)(x_2, f(x_2)) \in A.$$

That is, $(\alpha x_1 + (1 - \alpha)x_2, \alpha f(x_1) + (1 - \alpha)f(x_2)) \in A$. By the definition of A,

$$f(\alpha x_1 + (1 - \alpha)x_2) \geq \alpha f(x_1) + (1 - \alpha)f(x_2).$$

Hence f is concave. $\square$

Theorem 3.7. *Let $f : S \to \mathbb{R}$ be a concave function. Then, if S is open, f is continuous on S.*

Proof. Pick $x \in S$. Since S is open, there exists $r > 0$ such that $N_r(x) \subset S$ and its boundary $A = \{y : ||x - y|| = r\} \subset S$. Pick any $\{x_n\} \subset S$ such that $x_n \to x$. Then there exists N such that for all $n \geq N$, $||x_n - x|| < r$.

Then for all $n \geq N$, there exist $\theta_n \in (0, 1)$ and $z_n \in A$ such that $x_n = \theta_n x + (1 - \theta_n)z_n$. Since f is concave,

$$f(x_n) \geq \theta_n f(x) + (1 - \theta_n)f(z_n).$$

Taking limits on both sides,

$$\liminf_n f(x_n) \geq f(x).$$

Similarly, for all $n \geq N$, there exist $\lambda_n \in (0, 1)$ and $w_n \in A$ such that $x = \lambda_n x_n + (1 - \lambda_n)w_n$. Since f is concave,

$$f(x) \geq \lambda_n f(x_n) + (1 - \lambda_n)f(w_n).$$

Taking limits on both sides,

$$f(x) \geq \limsup_n f(x_n).$$

That is, $f(x_n)$ converges and $\lim_n f(x_n) = f(x)$. Since $\{x_n\}$ is picked arbitrarily, we know that $f(x)$ is continuous on S.

$\square$

Nevertheless, a function f is concave does not imply that it is always differentiable on the interior of its domain. The positive result on this is that if $f : (a, b) \to \mathbb{R}$ is concave, then it is differentiable at all but countably many points in (a, b). That is, it is differentiable almost everywhere (cf. Rockafeller, Convex Analysis, 1970).

Theorem 3.8. *Suppose $f : S \to \mathbb{R}$ is continuously differentiable. Then f is concave if and only if for all $x, y \in S$,*

$$f(y) \leq f(x) + Df(x) \cdot (y - x).$$

Proof. We only prove the necessity. Since f is concave, for any $\alpha \in [0, 1]$, $f(\alpha y + (1 - \alpha)x) \geq \alpha f(y) + (1 - \alpha)f(x)$. That is,

$$\begin{aligned} f(y) &\leq \frac{f(\alpha y + (1 - \alpha)x) - (1 - \alpha)f(x)}{\alpha} \\ &= f(x) + \frac{f(x + \alpha(y - x)) - f(x)}{\alpha}. \end{aligned}$$

By taking limit w.r.t. α on the right-hand-side, we have

$$f(y) \leq f(x) + Df(x) \cdot (y - x).$$

$\square$

The theorems below establishes the relationship between concavity and the second-order derivative of the function.

Theorem 3.9. *If $f : (a, b) \to \mathbb{R}$ is twice continuously differentiable, then f is concave if and only if $f''(x) \leq 0$ for all $x \in (a, b)$.*

Proof. ($\Leftarrow$) Suppose $f''(x) \le 0, \forall x \in (a,b)$. Then $f'(x)$ is non-increasing. For x, y such that $a < x < y < b$, pick $\alpha \in (0,1)$ and let $z = \alpha x + (1-\alpha)y$. Then

$$\begin{aligned} z - x &= (1-\alpha)(y-x), \\ y - z &= \alpha(y-x). \end{aligned}$$

Note that since $f'(x)$ is non-increasing,

$$\begin{aligned} f(z) - f(x) &= \int_x^z f'(t)dt \ge \int_x^z f'(z)dt = f'(z)(z-x), \\ f(y) - f(z) &= \int_z^y f'(t)dt \le \int_z^y f'(z)dt = f'(z)(y-z). \end{aligned}$$

Hence

$$\begin{aligned} f(z) &\ge f(x) + f'(z)(1-\alpha)(y-x), \\ f(z) &\ge f(y) - f'(z)\alpha(y-x). \end{aligned}$$

We have

$$\begin{aligned} f(z) &= \alpha f(z) + (1-\alpha) f(z) \\ &\ge \alpha[f(x) + f'(z)(1-\alpha)(y-x)] + (1-\alpha)[f(y) - f'(z)\alpha(y-x)] \\ &= \alpha f(x) + (1-\alpha) f(y). \end{aligned}$$

($\Rightarrow$) Suppose f is concave and instead there exists $x^0 \in (a,b)$ such that $f''(x^0) > 0$. Since $f''(x)$ is continuous, there exists $\varepsilon > 0$ such that $f''(x) > 0$ for all $x \in (x^0 - \varepsilon, x^0 + \varepsilon) \subset (a,b)$. Reverse the proof in the ($\Leftarrow$) part, we can find $x, y \in (x^0 - \varepsilon, x^0 + \varepsilon)$ and $\alpha \in (0,1)$ such that

$$f(ax + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y),$$

which means f is strictly convex on $(x^0 - \varepsilon, x^0 + \varepsilon)$; this contradicts with the concavity of f. $\square$

Definition 3.8. An $n \times n$ matrix $A = (a_{ij})_{i,j \le n}$ is symmetric if $a_{ij} = a_{ji}, \forall i, j \le n$. A symmetric A is negative semi-definite if for all vectors $z \in \mathbb{R}^n, z^T A z \le 0$. It is negative definite if for all $z \neq 0, z^T Az < 0$.

Theorem 3.10. *If C is an open convex subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ is twice continuously differentiable, then f is concave if and only if $D^2 f(x)$ is negative semi-definite for all $x \in C$.*

Proof. Part I. Pick arbitrary $y, z \in \mathbb{R}^n$ such that $\{\lambda : y + \lambda z \in C\}$ is nonempty. Let $g_{y,z}(\lambda) = f(\lambda(y+z) + (1-\lambda)y) = f(y + \lambda z)$. We will show that f is concave iff each $g_{y,z}(\lambda)$ is concave in λ.

First, Suppose f is concave. Pick λ, λ' such that $x := y + \lambda z \in C, x' := y + \lambda' z \in C$. Then $f(x) = g_{y,z}(\lambda)$ and $f(x') = g_{y,z}(\lambda')$. Furthermore,

$$\begin{aligned} f(\alpha x + (1-\alpha)x') &= f(y + (\alpha\lambda + (1-\alpha)\lambda')z) \\ &= g_{y,z}(\alpha\lambda + (1-\alpha)\lambda'). \end{aligned}$$

We see straightforwardly that the concavity of f implies that of each $g_{y,z}(\lambda)$.

Second, suppose $g_{y,z}(\lambda)$ is concave for each y, z. Pick $x, y \in C$ and let $z = x - y$. Then

$$\begin{aligned} y + \lambda z &= y + \lambda(x - y) \\ &= \lambda x + (1-\lambda)y. \end{aligned}$$

Therefore, for each $\lambda \in (0, 1)$,

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= g_{y,x-y}(\lambda) \\ &= g_{y,x-y}(\lambda \cdot 1 + (1-\lambda) \cdot 0) \\ &\geq \lambda g_{y,x-y}(1) + (1-\lambda) g_{y,x-y}(0) \\ &= \lambda f(y + (x - y)) + (1-\lambda) f(y + 0 \cdot z) \\ &= \lambda f(x) + (1-\lambda) f(y). \end{aligned}$$

Part II. Now we know that f is concave iff each $g_{y,z}(\lambda)$ is concave, that is, iff $[g_{y,z}(\lambda)]'' \leq 0$ for each pair of $y, z \in \mathbb{R}^n$. Let $x_0 = y + \lambda z$, then

$$\begin{aligned} g''_{y,z}(\lambda) &= \frac{\partial^2 f(y + \lambda z)}{\partial^2 \lambda} \\ &= z^T D^2 f(x_0) z. \end{aligned}$$

Hence $g''_{y,z}(\lambda) \leq 0$ for each pair y, z is equivalent to $D^2 f(x_0)$ being negative semi-definite for all $x_0 \in C$. □

Example 3.7. The Hicksian demand function is defined by $h((p,u) = \arg\min_{u(x)\geq u} p \cdot x$. Due to the Envelope theorem (to be studied), $h(p,u) = D_p e(p,u)$. Therefore, since $e(p,u)$ is concave in p, the Slutsky matrix $D_p h(p,u) = D_p^2 e(p,u)$ is negative semi-definite for all p, implying that the Hicksian demand satisfies the law of demand.

3.4 Quasi-concave functions

Definition 3.9. Let $f : S \to \mathbb{R}$, where $S \subset \mathbb{R}^n$ is convex. The function f is

(a) quasiconcave if $f(\alpha x + (1-\alpha)x') \geq \min\{f(x), f(x')\}$, for any $x, x' \in S$ and $\alpha \in (0,1)$, and is

(b) quasiconvex if $f(\alpha x + (1-\alpha)x') \leq \max\{f(x), f(x')\}$, for any $x, x' \in S$ and $\alpha \in (0,1)$.

The function f is strictly quasiconcave, or quasiconvex, respectively, if the inequality holds strict for all x, x', α. If f is quasiconcave, then its value at the average of two points is greater than or equal to the minimum of its values at the two points. Quasiconcave functions are strictly more general than concave functions. If a function is monotonic, then it is both quasiconcave and quasiconvex.

Example 3.8. $f(x) = e^x$ is quasiconcave, although it is strictly convex. The Cobb-Douglas function $f(x_1, x_2) = x_1^a x_2^b$ is quasiconcave if $a, b > 0$.

The main reason that we care about quasiconcave functions in economics is because of the following result, which intuitively states that a utility function is quasiconcave if and only if its indifference curves are boundaries of some convex set, and this captures diminishing marginal rate of substitution.

Theorem 3.11. *Let $U(c) = \{x \in S : f(x) \geq c\}$ be the upper contour set of f for level c. Then f is quasiconcave iff $U(c)$ is a convex set for all $c \in \mathbb{R}$.*

Proof. ($\Rightarrow$) Suppose f is quasiconcave. If $x, x' \in U(c)$, then $f(x) \geq c$ and $f(x') \geq c$. Thus for all $\alpha \in (0,1)$,

$$f(ax + (1-\alpha)x') \geq \min\{f(x), f(x')\} \geq c,$$

which implies $\alpha x + (1-\alpha)x' \in U(c)$.

($\Leftarrow$) Suppose $U(c)$ is a convex set for each $c \in \mathbb{R}$. For any $x, x' \in S$, obviously, $f(x), f(x') \geq \min\{f(x), f(x')\}$. Therefore, $x, x' \in U(\min\{f(x), f(x')\})$. Since $U(\min\{f(x), f(x')\})$ is a convex set, $\forall \alpha \in [0,1], \alpha x + (1-\alpha)x' \in U(\min\{f(x), f(x')\})$. That is, $\forall \alpha \in [0,1], f(\alpha x + (1-\alpha)x') \geq \min\{f(x), f(x')\}$.

$\square$

Similarly, let $L(c) = \{x \in S : f(x) \leq c\}$ be the lower contour set of f for level c. Then f is quasiconvex iff $L(c)$ is a convex set for all $c \in \mathbb{R}$.

Theorem 3.12. *Suppose $f : S \to \mathbb{R}$ is continuously differentiable. Then f is quasiconcave if and only if for all $x, y \in S$,*

$$f(y) \geq f(x) \Rightarrow Df(x) \cdot (y-x) \geq 0.$$

Proof. Again, we only prove for necessity. Since f is quasiconcave and $f(y) \geq f(x)$, for any $\alpha \in [0,1]$, $f(\alpha y + (1-\alpha)x) \geq f(x)$. That is

$$\frac{f(x + \alpha(y-x)) - f(x)}{\alpha} \geq 0.$$

By taking derivative w.r.t. α for the left-hand-side, we will have $Df(x) \cdot (y-x) \geq 0$.

$\square$

Example 3.9. Let $u : \mathbb{R}^2_+ \to \mathbb{R}$ denote a consumer's utility function. Then the boundary of the upper contour set, $u(x,y) = c$, is exactly the indifference curve of the consumer at utility level c. Thus $u(x,y)$ is quasiconcave guarantees that the indifference curve is of convex shape. That is, the function $y(x)$ that solves $u(x,y) = c$ is convex in x. This implies diminishing marginal rate of substitution, that is, it guarantees that

$$MRS_{xy} = -\frac{dy}{dx} = \frac{u_x(x,y)}{u_y(x,y)}$$

is decreasing in x. That is, the more of good x that you are consuming right now, the less of good y is needed to compensate you for your loss of 1 unit of good x.

3.5 A digression on ordinal and cardinal utility

Definition 3.10. For any nondecreasing function g on $\mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}$, the composition $g \circ u$ that maps $x \in \mathbb{R}^n$ to $g(u(x))$ is called a monotonic (nondecreasing) transformation of u.

Example 3.10. The utility functions $3x_1x_2 + 2, (x_1x_2)^2, (x_1x_2)^3 + x_1x_2, e^{x_1x_2}$ and $\ln x_1x_2$ are all monotonic transformations of $u(x_1, x_1) = x_1x_2$.

Intuitively, a preference relation is ordinal if it defines the relative order of the objects (e.g., prefer bundle (x_1, x_2) over bundle (y_1, y_2)), and it is cardinal if it also defines the intensity of comparison (e.g., bundle (x_1, x_2) generates 5 units more utility than bundle (y_1, y_2)). Formally,

Definition 3.11. A property of functions is called ordinal if whenever a function u has this property, for any monotonic transformation g, $g \circ u$ also has this property. Otherwise, it is called cardinal.

Claim 3.13. *Concavity is a cardinal property.*

This can easily be seen from the following example.

Example 3.11. $u(x) = \sqrt{x}$ is concave on $[0, \infty)$, but the monotonic transformation of it, $g(u(x)) = (\sqrt{x})^4 = x^2$, is strictly convex.

Claim 3.14. *Quasiconcavity is an ordinal property.*

Proof. This is immediate. Since g is nondecreasing and f is quasiconcave, $\forall \alpha \in (0, 1)$ and $x, y \in S$,

$$\begin{aligned} g(f(\alpha x + (1-\alpha)y)) &\geq g(\min\{f(x, f(y))\}) \\ &= \min\{g(f(x)), g(f(y)). \end{aligned}$$

$\square$

4 Optimization I (Equality constraints)

4.1 Unconstrained optimization

For any set $D \subset X$, let $\operatorname{int} D = \{x \in D : \exists r > 0, \text{ s.t., } N_r(x) \subset D\}$ be the set of interior points of D. If $Df(x^*) = 0$ for some $x^* \in \operatorname{int} D$, then x^* is said to be a critical point of f on D.

Theorem 4.1. *Suppose $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and $x \in \operatorname{int} D$. Then if f has a local maximum (or minimum) at x and $Df(x)$ exists, then $Df(x) = 0$.*

It is easy to see that for a function to achieve local maximum at x, then the value of f should not increase if we vary x a little in any direction h. That is, we need $Df(x; h) = Df(x) \cdot h = 0$ for any h, which happens only if $Df(x) = 0$. However, FOC is necessary but not sufficient for local optimality.

Example 4.1. Consider $f(x) = x^3$. Although $f'(0) = 0, f$ does not have local max or min at 0.

Example 4.2 (Saddle point)**.** Consider $f(x, y) = x^2 - y^2$. Then $Df(0, 0) = 0$, but f does not have local max or min at $(0, 0)$.

We need concavity or local concavity to ensure that FOC is also sufficient for global or local maximum. For a concave function, local maximum and global maximum coincide, and if a function is strictly concave, the maximizer is unique.

Theorem 4.2. *Let $D \subset \mathbb{R}^n$ be convex, and $f : D \to \mathbb{R}$ be a concave and differentiable function on D. Then, x is an unconstrained maximum of f on D if and only if $Df(x) = 0$.*

If we replace the concavity condition in the theorem with strict quasi-concavity, the theorem still holds.

Proof. Only need to prove ($\Leftarrow$). Since f is concave, for any $x, y \in D$ and $\alpha \in (0, 1]$

$$\begin{aligned} \alpha f(y) + (1 - \alpha) f(x) &\leq f(\alpha y + (1 - \alpha) x) \\ f(y) - f(x) &\leq \frac{f(x + \alpha(y - x)) - f(x)}{\alpha}. \end{aligned}$$

Take the limit of the right hand side as $\alpha \to 0$, the inequality transforms to

$$f(y) - f(x) \leq Df(x) \cdot (y - x).$$

If $Df(x) = 0, f(x) \geq f(y), \forall y \in D$.

$\square$

Example 4.3. Let $f(x, y) = x^4 + 2y^2$. There is a unique critical point $(0, 0)$. Since f is strictly convex, $(0, 0)$ is the unique global minimizer of f.

Similarly, if $f(x^*) = 0$ and f is locally concave around x^*, then x is a local maximum of f. As usual, to check f is locally concave around x^*, you need to check that $D^2 f(x)$ is negative semidefinite for all x in some neighborhood of x^*. Also, if $f \in C^2$ and we already know that f has a local maximum at x^*, then we know that $D^2 f(x^*)$ is negative semidefinite. This is because otherwise there must be $z \in \mathbb{R}^n$ such that $z \cdot D^2 f(x^*) z > 0$, and $f \in C^2$ implies that f is strictly convex around x^* in some direction, so moving away from x^* in that direction or the reverse direction strictly increases $f(x)$.

For completeness, in below we present methods used to identify the definiteness of a symmetric matrix. Please refer to Sundaram, Section 1.5.2 for details. In practice, such work and the second-order conditions are less important, because most often we will be dealing with well-behaved (e.g., concave or convex) functions.

We already defined that a symmetric matrix $A \in \mathbb{R}^{n \times n}$ is negative definite (semidefinite) if $z \cdot Az < (\leq) 0, \forall z \in \mathbb{R}^n$, and is positive definite (semidefinite) if $z \cdot Az > (\geq) 0, \forall z \in \mathbb{R}^n$. The easier way to check definiteness is to consider submatrices of A. Let A_k denote the $k \times k$ submatrix of A that consists of only the first k rows and k columns of A, and let B_k denote a $k \times k$ submatrix of A that consists of any k rows and the corresponding k columns (e.g., the second and fourth row and the second and fourth column form one B_2) of A. Let $|M|$ denote the determinant of a matrix M.

Theorem 4.3. *An $n \times n$ symmetric matrix A is*

(a) *negative definite iff $(-1)^k |A_k| > 0$ for all $1 \leq k \leq n$, and is negative semidefinite iff $(-1)^k |B_k| \geq 0$ for all $1 \leq k \leq n$ and all B_k;*

(b) *positive definite iff $|A_k| > 0$ for all $1 \leq k \leq n$, and is positive semidefinite iff $|B_k| \geq 0$ for all $1 \leq k \leq n$ and all B_k.*

Example 4.4. Consider $f(x,y) = x^3 + y^3 - 3xy$. The first order conditions for optima are

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases}.$$

Solving them gives us two critical points, $(0,0)$ and $(1,1)$. The Hessian matrix of f is

$$H(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}.$$

As $|H(0,0)| = -9 < 0, H(0,0)$ is neither negative semidefinite nor positive semidefinite, so $(0,0)$ is not a local optima. And as $f_{11}(1,1) = 6 > 0$ and $|H(1,1)| = 27 > 0, H(1,1)$ is positive definite. Hence $(1,1)$ is a strict local minimum.

4.2 Optimization with equality constraints

We first review the classical example of a utility maximization problem

$$\begin{aligned} &\max_{(x_1,x_2)\in\mathbb{R}^2} u(x_1,x_2) \\ &\text{s.t. } g(x_1,x_2) = p_1x_1 + p_2x_2 - I = 0. \end{aligned}$$

The arbitrage argument. Let $dx = (dx_1, dx_2), u_1 = \frac{\partial u}{\partial x_1}$ and $u_2 = \frac{\partial u}{\partial x_2}$, then $du = u(x+dx) - u(x) = u_1dx_1 + u_2dx_2$ is the change in utility from changing the consumption of good 1 by dx_1 and the consumption of good 2 by dx_2. Now suppose the change of consumption is due to the reallocation of an amount dI of money from buying good 2 to buying good 1. Then $dx_1 = \frac{dI}{p_1}, dx_2 = -\frac{dI}{p_2}$, and $du = (\frac{u_1}{p_1} - \frac{u_2}{p_2})dI$. Intuitively, $\frac{u_1}{p_1}$ is the marginal utility from spending one more dollar on good 1. If $\frac{u_1}{p_1} - \frac{u_2}{p_2} > 0$, then $du > 0$. The reallocation is benefitable for the consumer. Similarly, if $\frac{u_1}{p_1} - \frac{u_2}{p_2} < 0$, it would be benefitable for the consumer to reallocate money from good 1 to good 2.

If $\bar{x}$ is a local maximum, then both operations should not be benefitable at $\bar{x}$, that is $\frac{u_1}{p_1} - \frac{u_2}{p_2} \leq 0$ and $\frac{u_1}{p_1} - \frac{u_1}{p_1} \geq 0$. Hence at the optimal bundle $\bar{x}$, $\frac{u_1(\bar{x})}{p_1} = \frac{u_2(\bar{x})}{p_2}$.

The tangency argument. Graphically, the consumer's utility is maximized at $\bar{x}$ if the indifference curve is tangent to the budget line $p_1x_1 + p_2x_2 = I$ at $\bar{x}$. That is, at $\bar{x}$, the marginal rate of substitution (MRS) satisfies $\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$. Let $c = u(\bar{x})$ and consider the indifference curve $u(x_1, x_2) = c$. From the implicit function theorem, $u_1 dx_1 + u_2 dx_2 = 0$. Therefore, $\frac{u_1(\bar{x})}{u_2(\bar{x})} = -\frac{dx_2}{dx_1} = \frac{p_1}{p_2}$. We obtain the same condition on optimality as the one obtained from the arbitrage argument.

The multiplier. Since $g(x_1, x_2) = p_1 x_1 + p_2 x_2 - I$, we have $\frac{\partial g}{\partial x_1} = p_1$, $\frac{\partial g}{\partial x_2} = p_2$. At the local maximum $\bar{x}$, let $\lambda = \frac{u_1}{p_1} = \frac{u_2}{p_2}$. Then $u_1 = \lambda \frac{\partial g}{\partial x_1}, u_2 = \lambda \frac{\partial g}{\partial x_1}$. That is, $\triangledown u = \lambda \nabla g$.

4.2.1 The theorem of Lagrange

Consider the following optimization problem with equality constraints:

$$\begin{aligned} &\max\nolimits_{x \in \mathbb{R}^n} f(x) \\ &\text{s.t. } g_1(x) = b_1 \\ &\qquad \cdots \\ &\qquad g_m(x) = b_m. \end{aligned}$$

where the objective function f and constraint functions $g_1, \ldots, g_m$ are real-valued continuously differentiable functions on $\mathbb{R}^n$. The constraint set, denoted by C, consists of $x \in \mathbb{R}^n$ that satisfies all constraints. We assume that $n \geq m$; if $m > n$ then in general C will be empty. A point $\bar{x} \in C$ is a local constrained maximizer of the above problem if $\bar{x}$ is locally optimal within some $N_\varepsilon(\bar{x}) \cap C$.

The $m \times n$ matrix

$$Dg(x) = \begin{pmatrix} \nabla g_1(x)^T \\ \vdots \\ \nabla g_m(x)^T \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1(x)}{\partial x_1} & \cdots & \frac{\partial g_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m(x)}{\partial x_1} & \cdots & \frac{\partial g_m(x)}{\partial x_n} \end{pmatrix}$$

is called the Jacobian matrix. We say that the constraints $g_1, g_2, \ldots, g_m$ satisfy nondegenerate constraint qualification (NDCQ) at $\bar{x} \in \mathbb{R}^n$ if the matrix $Dg(\bar{x})$ has full rank m.

Theorem 4.4 (Lagrange)**.** *Suppose that the objective and constraint functions of the problem above are differentiable, $\bar{x} \in C$ is a local constrained maximizer and NDCQ is satisfied at $\bar{x}$. Then there are real numbers $\lambda_1, \ldots \lambda_m$, one for each constraint, such that*

$$Df(\bar{x}) = \lambda_1 Dg_1(\bar{x}) + \cdots + \lambda_m Dg_m(\bar{x}).$$

The λ's are referred to as the Lagrange multipliers. The Lagrangean of the problem is defined as

$$L(x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_m)=f(x)+[\lambda_1(b_1-g_1(x))+\cdots+\lambda_m(b_m-g_m(x)].$$

The theorem above basically says that if $\bar{x}$ is a local constrained maximizer of the orginal problem that satisfies NDCQ, then there exist λ's such that $(\bar{x}_1\ldots,\bar{x}_n,\lambda_1,\ldots,\lambda_m)$ is a critical point of the Lagrangean.

Proof. Since $Dg(\bar{x})$ has full rank m, there are m columns out of the n columns that are independent with each other. Without loss of generality, assume these columns correspond to $x_1,\ldots,x_m$. That is, at $\bar{x}$,

$$\begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{vmatrix} \neq 0.$$

The implicit function theorem ensures that given the equality constraints, $x_1,\ldots,x_m$ can be solved as functions of $x_{m+1},\ldots,x_n$ around $\bar{x}=(\bar{x}_1,\ldots,\bar{x}_n)$. Therefore, if $\bar{x}$ is a local constrained maximizer of $f(x)$, then $(\bar{x}_{m+1},\ldots,\bar{x}_n)$ is a local maximizer of the unconstrained problem

$$\max f(x_1(x_{m+1},\ldots,x_n),\ldots,x_m(x_{m+1}\ldots,x_n),x_{m+1},\ldots,x_n).$$

The FOCs of this problem are

$$(\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_m})\begin{pmatrix} \frac{\partial x_1}{\partial x_j} \\ \vdots \\ \frac{\partial x_m}{\partial x_j} \end{pmatrix}+\frac{\partial f}{\partial x_j}=0, \forall j=m+1,\ldots,n.$$

Due to the implicit function theorem,

$$\begin{pmatrix} \frac{\partial x_1}{\partial x_j} \\ \vdots \\ \frac{\partial x_m}{\partial x_j} \end{pmatrix} = -\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x_j} \\ \vdots \\ \frac{\partial g_m}{\partial x_j} \end{pmatrix}, \forall j=m+1,\ldots,n.$$

Let

$$(\lambda_1, \ldots, \lambda_m) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}) \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_m} \end{pmatrix}^{-1}$$

Therefore, the FOCs can be rearranged to generate

$$\frac{\partial f}{\partial x_j} = (\lambda_1, \ldots, \lambda_m) \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x_j} \\ \vdots \\ \frac{\partial g_m}{\partial x_j} \end{pmatrix}, \forall j = m+1, \ldots, n,$$

and simply by the definition of the λ's, for $i = 1, \ldots, m$

$$\frac{\partial f}{\partial x_i} = (\lambda_1, \ldots, \lambda_m) \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x_i} \\ \vdots \\ \frac{\partial g_m}{\partial x_i} \end{pmatrix}.$$

□

The "cookbook" procedure for solving an optimization problem (Sundaram, Section 5.4) is to first set up the Lagrangean, derive all the first-order conditions with respect to all x and λ's, and then solve for critical points from the FOCs. After that, compare the critical points and pick out the maximum.

Example 4.5. Consider the problem $\max_{(x,y)} x^2 - y^2$, subject to $x^2 + y^2 = 1$.

Solution: The Lagrangean of this problem is

$$L(x, y, \lambda) = x^2 - y^2 + \lambda(1 - x^2 - y^2).$$

The FOCs are

$$\begin{aligned} \frac{\partial L(x,y,\lambda)}{\partial x} &= 2x - 2\lambda x = 0, \\ \frac{\partial L(x,y,\lambda)}{\partial y} &= -2y - 2\lambda y = 0, \\ \frac{\partial L(x,y,\lambda)}{\partial \lambda} &= 1 - x^2 - y^2 = 0. \end{aligned}$$

Solving them together, we obtain four critical points

$$(x, y, \lambda) = (1, 0, 1), (-1, 0, 1), (0, 1, -1), (0, -1, -1).$$

By comparing values of f at these critical points, we see that f obtains global maximum at the first two points and obtains global minimum at the latter two points.

For most problems, the "cookbook" procedure of Lagrangean works well; in some rare cases, it could be problematic. When the optimal solution does not satisfy NDCQ, the critical points solved from the FOCs may not include the optimal solution. And in principle, even if the optimal solution satisfies NDCQ, the FOCs are only necessary for optimization: not all critical points that solve the FOCs are optimal; in fact, they may not even be locally optimal. Furthermore, when a critical point is optimal, it could be either maximal or minimal. Second-order conditions on the Lagrangean is useful in identifying whether $L(x, \lambda)$ is locally concave or convex around the critical points and within the constraint set, and hence useful to check whether a critical point is a locally maximal or minimal. See Sundaram (section 5) for examples and discussions on these issues.

4.2.2 Envelope theorem

Consider first the producer's problem. Suppose that f is the production function, p is the price of the output, x is input and w is its price. The profit function is then $\pi(x, p, w) = pf(x) - wx$. Let $v(p, w)$ denote the optimal profit given p, w. Suppose $\bar{x}(p, w)$ maximizes $\pi(x, p, w)$, then the optimal profit is $\pi(\bar{x}(p, w), p, w)$. From FOC of the profit maximization, $pf'(\bar{x}(p, w)) - w = 0$. What is marginal effect of p on the optimal profit?

$$\begin{aligned}
\frac{\partial \pi(\bar{x}(p, w), p, w)}{\partial p} &= \frac{\partial [pf(\bar{x}(p, w)) - w\bar{x}(p, w)]}{\partial p} \\
&= \frac{\partial \pi}{\partial p} + \frac{\partial \pi}{\partial x}\frac{\partial \bar{x}}{\partial p} \\
&= f(\bar{x}(p, w)) + (pf'(\bar{x}(p, w)) - w)\frac{d\bar{x}(p, w)}{dp} \\
&= f(\bar{x}(p, w)).
\end{aligned}$$

There are two effects that a change in p could have on $\pi(\bar{x}(p, w), p, w)$. One is the direct effect $\partial\pi/\partial p$, and the other is the indirect effect $(\partial\pi/\partial\bar{x})\cdot(\partial\bar{x}/\partial p)$, which takes place through $\bar{x}(p,w)$. The indirect effect is zero, because $\frac{\partial\pi}{\partial\bar{x}} = 0$: at the optimum $\bar{x}$, small changes in x won't affect π.

We now consider the envelope theorem for optimization with equality constraints. Suppose $q = (q_1, \dots, q_K)$ is the vector of parameters of the optimization problem

$$v(q) = \max_{x\in\mathbb{R}^n} f(x,q) \text{ such that } g(x,q) = 0,$$

where the constraint can be viewed as a vector of constraints (in that case, λ will be a vector of multipliers). The Lagrangean of this problem is

$$L(x,q;\lambda) = f(x,q) + \lambda\cdot[0 - g(x,q)].$$

Theorem 4.5 (Envelope theorem)**.** *Assume $v(q)$ is differentiable in q and λ is the multiplier associated with the maximizer $\bar{x}(q)$. Then*

$$\frac{\partial v(q)}{\partial q_k} = \frac{\partial L(\bar{x},q;\lambda)}{\partial q_k}, \forall 1\le k\le K.$$

Proof. Since $g(\bar{x}(q),q) = 0$,

$$\frac{\partial g(\bar{x}(q),q)}{\partial q_k} = \sum_{i=1}^{n}\frac{\partial g(\bar{x}(q),q)}{\partial x_i}\frac{\partial \bar{x}_i(q)}{\partial q_k} + \frac{\partial g(\bar{x}(q),q)}{\partial q_k} = 0.$$

Therefore,

$$\begin{aligned}
\frac{\partial v(q)}{\partial q_k} &= \frac{\partial f(\bar{x}(q),q)}{\partial q_k} \\
&= \sum_{i=1}^{n}\frac{\partial f(\bar{x}(q),q)}{\partial x_i}\frac{\partial \bar{x}_i(q)}{\partial q_k} + \frac{\partial f(\bar{x}(q),q)}{\partial q_k} \\
&= \sum_{i=1}^{n}\lambda\frac{\partial g(\bar{x}(q),q)}{\partial x_i}\frac{\partial \bar{x}_i(q)}{\partial q_k} + \frac{\partial f(\bar{x}(q),q)}{\partial q_k} \\
&= \lambda\cdot(-\frac{\partial g(\bar{x}(q),q)}{\partial q_k}) + \frac{\partial f(\bar{x}(q),q)}{\partial q_k} \\
&= \frac{\partial L(\bar{x},q;\lambda)}{\partial q_k}.
\end{aligned}$$

We used FOC to obtain the third equality and used the formula on $\partial g(\bar{x}(q), q)/\partial q_k$ to obtain the fourth equality.

$\square$

4.2.3 The Lagrangean multiplier

Consider $\max_{x \in \mathbb{R}^n} f(x)$, subject to $g_1(x) = b_1, \ldots, g_m(x) = b_m$, and focus on the parameter vector $b = (b_1, \ldots, b_m)$. Suppose $\bar{x}(b)$ is a maximizer that corresponds to b, then $v(b) = f(\bar{x}(b))$. Due to the envelope theorem,

$$\begin{aligned} \frac{\partial v(b)}{\partial b_k} &= \frac{\partial [f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x))]}{\partial b_k} \\ &= \lambda_k. \end{aligned}$$

This result implies that λ_k is exactly the marginal effect of relaxing the constraint $g_k(x) = b_k$ on the value $v(b)$ of the problem. In particular, recall the simple utility maximization problem where $v(p, I) = \max u(x)$ subject to $p \cdot x = I$. We have

$$\frac{\partial v(p, I)}{\partial I} = \lambda = \frac{u_1(\bar{x})}{p_1} = \frac{u_2(\bar{x})}{p_2}.$$

So, λ describes the marginal utility of income–the increment of utility from spending one more dollar. It is also called the "shadow price" from relaxing the (budget) constraint.

5 Optimization II (Inequality constraints)

5.1 Utility maximization revisited

Let's revisit the utility maximization problem.

$$\begin{aligned}&\max_{(x_1,x_2)\in\mathbb{R}^2} u(x_1,x_2)\\ &\text{s.t. } g(x_1,x_2)=p_1x_1+p_2x_2-I\le 0.\\ &\quad x_1\ge 0, x_2\ge 0.\end{aligned}$$

Note that we have rewritten the budget constraint as an inequality constraint and have added the nonnegativity of quantities x_1 and x_2 into the constraints. Starting with any consumption boundle $x=(x_1,x_2)$, if we reallocate dI units of money from buying good 2 to good 1, then $du=(\frac{u_1(x)}{p_1}-\frac{u_2(x)}{p_2})dI$. This operation is always benefitable at x as long as $\frac{u_1(x)}{p_1}-\frac{u_2(x)}{p_2}>0$.

What if this condition still holds at $(\frac{I}{p_1},0)$, that is, when you have already spent all your money on good 1 (happen when the utility function takes specific forms)? Due to the nonnegativity constraint on x_2, you cannot decrease x_2 any more. As a result, the optimal bundle is $x=(\frac{I}{p_1},0)$; it is not in the interior of the constraint set and is hence called a *corner solution.*

Example 5.1. Solve $\max_{(x_1,x_2)} x_1(x_2+3)$, s.t. $x_1+x_2\le 2$. The optimal solution (the tangent point) is $(x_1^*,x_2^*)=(5/2,-1/2)$. With nonnegative constraints on x, $(x_1^*,x_2^*)=(2,0)$.

5.2 Kuhn-Tucker conditions

For simplicity, we focus on the case that there are only inequality constraints. Consider the following optimization problem with m inequality constraints:

$$\begin{aligned}&\max_{x\in\mathbb{R}^n} f(x)\\ &\text{s.t. } g_1(x)\le b_1\\ &\qquad\cdots\\ &\qquad g_m(x)\le b_m\end{aligned}$$

We assume that the functions f and the g_i's are continuously differentiable. Let $C=\{x\in\mathbb{R}^n : g_i(x)\le b_i, \forall 1\le i\le m\}$ be the constraint set, which contains all points x that satisfy the inequality constraints. And local and global constrained optima are defined as usual. We say that constraint i is binding at x if $g_i(x) = b_i$. Otherwise, we say that it is slack at x.

The nondegenerate constraint qualification (NDCQ) is satisfied at $\bar{x} \in C$ if the gradients of binding constraints at $\bar{x}$ are independent, that is, if the set of vectors $\{\nabla g_i(\bar{x})\}_{i \in I(\bar{x})}$ are linearly independent, where $I(\bar{x}) = \{i : g_i(\bar{x}) = b_i\}$.

Theorem 5.1 (Kuhn-Tucker)**.** *Suppose that $\bar{x} \in C$ is a local maximizer of the above problem. Assume also that NDCQ is satisfied at $\bar{x}$. Then there are **nonnegative** real numbers $\lambda_1, \ldots, \lambda_m$, one for each inequality constraint, such that*

1. $\frac{\partial f(\bar{x})}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(\bar{x})}{\partial x_j} = 0, \forall 1 \leq j \leq n;$
2. $g_i(\bar{x}) \leq b_i, \lambda_i \geq 0,$ *and* $\lambda_i(b_i - g_i(\bar{x})) = 0,$ *for each* $1 \leq i \leq m.$

The Lagrangean of the problem is defined as

$$L(x, \lambda) = f(x) + \lambda_1(b_1 - g_1(x)) + \cdots + \lambda_m(b_m - g_m(x)).$$

Additional to the usual first-order conditions of the Lagrangean, the Kuhn-Tucker conditions also require that for each constraint $i, \lambda_i(b_i - g_i(x)) = 0$. This condition is called complementary slackness: if one of λ_i and constraint i is slack, then the other must be binding.

When there are both equality and inequality constraints, we set up the Lagrangean for all constraints, while require nonnegativity of the multiplier and complementary slackness only for inequality constraints. In below, we still assume that there are only inequality constraints.

By constructing the Lagrangean $L(x, \lambda)$, again we are transforming the original constrained optimization problem to the unconstrained optimization of the Lagrangean, so that we can look for solutions of the original problem among critical points of the Lagrangean. With only equality constraints (i.e., $g_i(x) = b_i, \forall i$), for each critical point $(\bar{x}, \lambda)$, the interpretation of the associated λ's is that they are penalty (punishment) weights on the constraints around $\bar{x}$: the penalty is λ_i per unit change of b_i, which equals exactly the marginal value of b_i to $v(b)$. Moreoever, depending on whether we write the constraint as $g_i(x) = b_i$ or $-g_i(x) = -b_i$, the sign of λ_i can be positive or negative.

However, relaxing or violating any inequality constraint can only have nonnegative effect on the optimal value of $f(x)$, since doing so enlarges the constrained set. If at a constrained local maximizer $\bar{x}$ constraint i is slack, then $f(x)$ is locally maximized at the interior of constraint i. As a result, any small deviation from $\bar{x}$ that does not violate constraint i is not benefitable and need not be associated with any penalty; that is, $\lambda_i = 0$. Conversely, if at some constrained local maximizer $\bar{x}$, relaxing or violating constraint i a little has strictly positive effect on $f(x)$, that is, if $\lambda_i > 0$, then it has to be the case that constraint i is binding at $\bar{x}$ (in this case, we say that $\bar{x}$ is a corner solution, since it is maximized at the corner).

In order for λ to capture the nonnegative effect of enlarging the constraint set, we should always be careful when formulating the Lagrangean with inequality constraints. When you see any constraint $g_i(x) \leq b_i$, you should add it to the Lagrangean as a term $\lambda_i(b_i - g_i(x))$, instead of as $\lambda_i(g_i(x) - b_i)$. And when you have a minimization problem, it is also helpful to reformulate it as the problem of maximizing $-f(x)$ before solving it.

Proof of Kuhn-Tucker's theorem. Suppose $\bar{x}$ is a local maximizer of $f(x)$ and NDCQ is satisfied at $\bar{x}$. We first show that there exist $\{\lambda_i\}_{i \in I(\bar{x})}$ for constraints that bind at $\bar{x}$ such that

$$Df(\bar{x}) = \sum_{i \in I(\bar{x})} \lambda_i Dg_i(\bar{x}),$$

and then show that these λ's are nonnegative. Suppose instead such λ's do not exist. Then since the vectors $\{Dg_i(\bar{x})\}_{i \in I(\bar{x})}$ are indepdent due to NDCQ, the vectors $\{Dg_i(\bar{x})\}_{i \in I(\bar{x})} \cup \{Df(\bar{x})\}$ must also be independent. Therefore, we can find a vector $z \in \mathbb{R}^n$, s.t.

$$Df(\bar{x}) \cdot z = 1, \text{ and } Dg_i(\bar{x}) \cdot z = -1, \forall i \in I(\bar{x}).$$

For $\varepsilon > 0$ small enough, due to Taylor's theorem and the continuity of $Df(x)$ and $Dg_i(x)$'s, for each $i \in I(\bar{x})$,

$$\begin{aligned} g_i(\bar{x} + \varepsilon z) &= g_i(\bar{x}) + Dg_i(\bar{x} + \xi z) \cdot \varepsilon z < g_i(\bar{x}) = b_i, \\ f(\bar{x} + \varepsilon z) &= f(\bar{x}) + Df(\bar{x} + \xi' z) \cdot \varepsilon z > f(\bar{x}), \end{aligned}$$

where ξ and ξ' are some numbers that satisfy $0 < \xi, \xi' < \varepsilon$. So by moving away from $\bar{x}$ by εz, all slack constraints are still slack, and all binding constraints become slack. Nevertheless, the value of f strictly increases. This contradicts with the local optimality of $f(x)$ at $\bar{x}$.

Now suppose for some $i \in I(\bar{x})$, $\lambda_i < 0$. (Intuitively, this means that relaxing constraint i has negative effect on $f(x)$.) Let $M < 0$ be some number that satisfies $\lambda_i M > \sum\limits_{i' \in I(\bar{x}), i' \neq i} \lambda_{i'}$. Since $\{Dg_{i'}(\bar{x})\}_{i' \in I(\bar{x})}$ are independent, there exists $z \in \mathbb{R}^n$ such that $Dg_i(\bar{x}) \cdot z = M$ while $Dg_{i'}(\bar{x}) \cdot z = -1, \forall i' \neq i, i' \in I(\bar{x})$. Therefore,

$$
\begin{aligned}
f(\bar{x} + \varepsilon z) &= f(\bar{x}) + Df(\bar{x}) \cdot \varepsilon z + o(\varepsilon) \\
&= f(\bar{x}) + \varepsilon \sum_{i \in I(\bar{x})} \lambda_i Dg_i(\bar{x}) \cdot z + o(\varepsilon) \\
&= f(\bar{x}) + \varepsilon[\lambda_i M - \sum_{i' \in I(\bar{x}), i \neq i'} \lambda_{i'}] + o(\varepsilon).
\end{aligned}
$$

Since the second term is strictly positive and the third term is just the remainder of the Taylor expansion, $f(\bar{x} + \varepsilon z) > f(\bar{x})$. Again this contradicts with the local optimality of $f(x)$ at $\bar{x}$.

Lastly, for each constraint j that is slack at $\bar{x}$, let $\lambda_j = 0$. The existence of $\lambda_1, \ldots, \lambda_m$ that satisfy the conditions is thus proved.

□

This proof is taken from Kreps' book, *Microeconomic Foundations I.* It can be further shortened by using the Farkas' Lemma (as in Lin Zhou's lecture notes) or an argument that captures the idea of the lemma (as in MWG, Theorem M.K.2).

In below we solve the previous example as an illustration of the K-T method.

Example 5.2. Solve $\max_{(x_1,x_2)} x_1(x_2 + 3)$, s.t. $x_1 + x_2 \leq 2, x_1, x_2 \geq 0$.

Solution: Let

$$L(x_1, x_2, \lambda, \mu_1, \mu_2) = x_1(x_2 + 3) + \lambda(2 - x_1 - x_2) + \mu_1 x_1 + \mu_2 x_2.$$

The FOCs, complementary-slackness, and nonnegativity conditions are

(1) $\frac{\partial L}{\partial x_1} = x_2 + 3 - \lambda + \mu_1 = 0;$

(2) $\frac{\partial L}{\partial x_2} = x_1 - \lambda + \mu_2 = 0;$

(3) $\lambda(2 - x_1 - x_2) = 0, \lambda \geq 0, 2 - x_1 - x_2 \geq 0$;

(4) $\mu_1 x_1 = 0, \mu_1 \geq 0, x_1 \geq 0$;

(5) $\mu_2 x_2 = 0, \mu_2 \geq 0, x_2 \geq 0$.

We solve this inequality system by discussion:

Case 1 Suppose $\lambda = 0$. Then by (1), $x_2 + 3 + \mu_1 = 0$. Since $x_2 \geq 0$ and $\mu_1 \geq 0$, this is not possible.

Therefore, $\lambda > 0$. By (3), $x_1 + x_2 = 2$.

Case 2 Given $\lambda > 0$, suppose $\mu_1 = 0$. Then by (1) and (2), $\lambda = x_2 + 3 = x_1 + \mu_2$. Together with $x_1 + x_2 = 2$, we have $\mu_2 = 2x_2 + 1 > 0$ and hence by (5), $x_2 = 0$. Consequently, $\mu_2 = 1, x_1 = 2, \lambda = 3$, and $\mu_1 = 0$. We have a solution $(x_1, x_2, \lambda, \mu_1, \mu_2) = (2, 0, 3, 0, 1)$.

Case 3 Given $\lambda > 0$, suppose $\mu_1 > 0$. Then $x_1 = 0$. By (2), $\mu_2 = \lambda > 0$; hence $x_2 = 0$. This contradicts with $x_1 + x_2 = 2$.

We have exhausted all possibilities (double check for this). The only solution is $(x_1^*, x_2^*) = (2, 0)$, which is a corner solution.

For most of the time, the Kuhn-Tucker method works well, but as usual, in rare cases, it may fail. It may fail if the global optimum does not exist, or if an optimum exists but the constraint qualification is not satisfied at the optimum.

Example 5.3. Consider $\max_{x,y} -(x^2 + y^2)$, subject to $(x-1)^3 - y^2 \geq 0$. By observation, f is maximized at $(\bar{x}, \bar{y}) = (1, 0)$. However, here $g(x,y) = y^2 - (x-1)^3$ and $Dg(\bar{x}, \bar{y}) = (0, 0)$, which is not of full rank. So NDCQ fails at $(\bar{x}, \bar{y})$.

Furthermore, there exists no λ such that $Df(\bar{x}, \bar{y}) - \lambda Dg(\bar{x}, \bar{y}) = (0, 0)$. So the Kuhn-Tucker conditions fail to identify the maximizer of this problem.

Moreover, even when NDCQ is satisfied, the Kuhn-Tucker conditions are only necessary for optimality: critical points that satisfy these conditions may not be local optimum. We have the following theorem on the sufficiency of the Kuhn-Tucker conditions.

Theorem 5.2 (Sufficiency). *Suppose $f(x)$ is concave, $g_i(x)$ is* quasiconvex *for each $1 \leq i \leq m$, and $\bar{x}$ satisfies all conditions in the Kuhn-Tucker theorem, then $\bar{x}$ maximizes $f(x)$ over the constraint set C.*

Proof. Consider any constraint i that is binding at $\bar{x}$. For any $x' \in C, g_i(x') \leq b_i = g_i(\bar{x})$. Since $g_i(\bar{x})$ is quasiconvex, for any $\alpha \in [0,1], g_i(\alpha\bar{x} + (1-\alpha)x') \leq g_i(\bar{x})$. We have

$$
\begin{aligned}
0 &\geq \lim_{\alpha \to 0} \frac{g_i(\bar{x} + \alpha(x' - \bar{x})) - g_i(\bar{x})}{\alpha} \\
&= Dg_i(\bar{x}) \cdot (x' - \bar{x}).
\end{aligned}
$$

Since $f(x)$ is concave,

$$
\begin{aligned}
f(x') &\leq f(\bar{x}) + Df(\bar{x}) \cdot (x' - \bar{x}) \\
&= f(\bar{x}) + \Big(\sum_{i \in I(\bar{x})} \lambda_i Dg_i(\bar{x})\Big) \cdot (x' - \bar{x}) \\
&\leq f(\bar{x}),
\end{aligned}
$$

where the last inequality is due to the nonnegativity of the λ's. We can now conclude that $\bar{x}$ maximizes $f(x)$.

□

We have the following remarks on this theorem:

1. We only need $f(x)$ to be pseudo-concave. A function $f(x)$ is pseudo-concave if $Df(\bar{x})(x' - \bar{x}) \leq 0$ implies $f(x') \leq f(\bar{x})$; this definition directly implies its sufficiency for local maximum (check with the proof above). Moreover, $f(x)$ is pseudo-concave iff it is quasiconcave and $Df(\bar{x}) = 0$ implies $f(x)$ attains global maximum at $\bar{x}$ (counterexample, $f(x) = x^3$).

2. When $f(x)$ is concave, NDCQ is not necessary anymore. Instead, it can be replaced by the Slater's condition, which requires that there exists $x \in C$ such that $g_i(x) < b_i, \forall 1 \leq i \leq m$. (See Sundaram, p188.)

6 The real field and metric spaces

6.1 Irrationality of $\sqrt{2}$

Theorem 6.1. *There is no rational number whose square is* 2.

Proof. Suppose not, then there must be integers p and q without common factors such that

$$(\frac{p}{q})^2 = 2.$$

Therefore

$$p^2 = 2q^2.$$

Now note that p must be an even number, because the square of an odd number is odd. Hence $p = 2r$ for some integer r, and

$$q^2 = 2r^2.$$

Similarly, we can see that q must also be an even number, and p and q have a common factor 2. Contradiction. □

This result can be generalized to any prime number p.

6.2 Definition of $\mathbb{R}$

First, $\mathbb{R}$ is a set containing $\mathbb{Q}$. We also assume that $\mathbb{R}$ is an ordered field, which contains $\mathbb{Q}$ as a subfield. The most distinctive assumption about the real number system is that it permits no gaps. In other words, it is complete. This assumption is formulated by the Axiom of Completeness.

Definition 6.1. A set $A \subset \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an upper bound for A.

A real number s is the least upper bound (supremum) of A, denoted as $\sup A$, if

(a) s is an upper bound for A;

(b) if b is any upper bound for A, then $s \leq b$.

Lower bounds and the greatest lower bound (infimum, $\inf A$) for A can be defined similarly.

Example 6.1. $\sup(0,1) = \sup[0,1] = 1, \inf(0,1) = \inf[0,1] = 0$. For $A = \{\frac{1}{n} : n \in \mathbb{N}\}, \sup A = 1, \inf A = 0$.

Axiom 6.2 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound.*

Claim 6.3. *The set of rational numbers $\mathbb{Q}$ does not satisfy the Axiom of Completeness.*

Proof. It is sufficient to show that the least upper bound of $S = \{r \in \mathbb{Q} : r^2 < 2\}$ does not belong to $\mathbb{Q}$. Let $a = \sup S$. Suppose instead $a \in \mathbb{Q}$. We know immediately that $a^2 \neq 2$. Define $b \in \mathbb{Q}$ as

$$b = a - \frac{a^2 - 2}{a + 2} = \frac{2a + 2}{a + 2}.$$

Then

$$b^2 - 2 = \frac{2(a^2 - 2)}{(a + 2)^2}.$$

If $a^2 < 2$, then $b > a$ and $b^2 < 2$. This contradicts with $a = \sup S$. Similarly, if $a^2 > 2$, then $b < a$ and $b^2 > 2$. This also contradicts with $a = \sup S$. □

Definition 6.2. A real number a_0 is a maximum (minimum) of the set A if $a_0 \in A$ and $a_0 \geq a$ ($a_0 \leq a$) for all $a \in A$.

The example below illustrates the difference between supremum (infimum) and maximum (minimum).

Example 6.2. While $\max[0,1] = 1$ and $\min[0,1] = 0$, $\max(0,1)$ and $\min(0,1)$ do not exist.

Theorem 6.4. *Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subset \mathbb{R}$. Then, $s = \sup A$ if and only if, for every $\varepsilon > 0$, there exists and element $a \in A$ such that $s - \varepsilon < a$.*

Proof. ($\Rightarrow$) Suppose instead $s = \sup A$ but there exists $\varepsilon > 0$ such that $s - \varepsilon \geq a, \forall a \in A$. Then $s - \varepsilon$ would be an upper bound of A that is smaller than s. This contradicts with $s = \sup A$.

($\Leftarrow$) By assumption, suppose s is not the supremum, then $s > \sup A$. Pick ε such that $0 < \varepsilon < s - \sup A$, then $s - \varepsilon > \sup A \geq a$, for all $a \in A$. Contradicts. □

6.3 Metric spaces

Definition 6.3. A set X, whose elements we shall call points, is said to be a metric space if with any two points p and q of X there is associated a real number $d(p, q)$, called the distance from p to q, such that

(a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;

(b) $d(p, q) = d(q, p)$;

(c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these properties is called a distance function, or a metric.

The most familiar metric space is the Euclidean space $\mathbb{R}^k$, where the distance between two vectors x and y is defined as $d(x, y) = ||x - y||$.

Lemma 6.5. *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} \cdot \mathbf{y}| \leq ||\mathbf{x}||||\mathbf{y}||$, *or equivalently,*

$$|\sum_{i=1}^{k} x_i y_i|^2 \leq \sum_{i=1}^{k} |x_i|^2 \sum_{i=1}^{k} |y_i|^2.$$

Proof. For any $t \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, we have $\mathbf{x} - t\mathbf{y} \in \mathbb{R}^k$. If there exists $t \in \mathbb{R}$ such that $\mathbf{x} = t\mathbf{y}$, it is straightforward that $|\mathbf{x} \cdot \mathbf{y}| = ||\mathbf{x}||||\mathbf{y}||$. If for all t, $\mathbf{x} \neq t\mathbf{y}$, then

$$(\mathbf{x} - t\mathbf{y})^2 > 0, \forall t,$$

that is

$$\mathbf{x}^2 - 2t\mathbf{x} \cdot \mathbf{y} + t^2\mathbf{y}^2 > 0, \forall t.$$

Note that the left hand side of the inequality is a quadratic function of t, so the discriminant of the equation must satisfy

$$(-2\mathbf{x} \cdot \mathbf{y})^2 - 4\mathbf{x}^2\mathbf{y}^2 < 0.$$

We have $|\mathbf{x} \cdot \mathbf{y}| < ||\mathbf{x}||||\mathbf{y}||$. $\square$

Recall that when $k = 2, |\cos\theta| = \frac{|\mathbf{x}\cdot\mathbf{y}|}{||\mathbf{x}||\,||\mathbf{y}||} \leq 1$, where θ is the angle between the vectors $\mathbf{x}$ and $\mathbf{y}$.

Theorem 6.6. *When the distance between x and y is induced by the norm, i.e., $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$, $\mathbb{R}^k$ is a metric space, .*

Proof. Part (a) and (b) are trivial. To prove (c), it is sufficient to show that

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||, \text{ for any } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k,$$

and then replace $\mathbf{x}$ by $\mathbf{x} - \mathbf{y}$ and $\mathbf{y}$ by $\mathbf{y} - \mathbf{z}$.

We have

$$\begin{aligned} ||\mathbf{x} + \mathbf{y}||^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x}^2 + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y}^2 \\ &\leq ||\mathbf{x}||^2 + 2||\mathbf{x}||\,||\mathbf{y}|| + ||\mathbf{y}||^2 \\ &= (||\mathbf{x}|| + ||\mathbf{y}||)^2. \end{aligned}$$

□

The induced metric

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_k - y_k)^2}$$

on $\mathbb{R}^k$ is called the Euclidean metric. Below are some other examples of metric spaces.

Example 6.3 (The taxicab distance)**.** Let

$$d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_k - y_k|.$$

Example 6.4 (Discrete metric)**.** Let X be an arbitrary set, for any $x, y \in X$, let

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Example 6.5 (sup norm)**.** Let X be the set of all real-valued continuous functions defined on a fixed closed interval $[a,b]$. For any $f,g\in X$, let

$$d(f,g)=\max_{a\leq s\leq b}|f(s)-g(s)|.$$

Example 6.6. Suppose A and B are two convex and compact subsets of $\mathbb{R}^n$. The Hausdorff distance (metric) between two sets A and B is defined as $d_H(A,B)=\max\{d(A,B),d(B,A)\}$, where $d(A,B)=\sup\{d(x,B),x\in A\}$ and $d(x,B)=\inf\{d(x,y),y\in B\}$.

6.4 Sequences and limits

6.4.1 Convergent sequences

Definition 6.4. Fix two sets A and B. If each $x\in A$ is associated with an element in B, denoted as $f(x)$, we say that f is a function (mapping) from A to B.

Let (X,d) be a metric space. A sequence in X is a function from $\mathbb{N}=(1,2,...)$ to X, denoted by $x_1,x_2,...,x_n,\ldots$, or simply $\{x_n\}$.

Definition 6.5. An ε-neighborhood of $x\in X$ is a set $N_\varepsilon(x)=\{y\in X: d(x,y)<\varepsilon\}$.

Definition 6.6. A sequence $\{x_n\}$ in X is said to converge if there is a point $x\in X$ with the property that for each $\varepsilon>0$, there is an integer $N\in\mathbb{N}$ such that $x_n\in N_\varepsilon(x)$ for all $n\geq N$. This is written as $\lim_{n\to\infty}x_n=x$.

By definition, whether $\{x_n\}\subset X$ converges depends both on which elements are in X and on the metric d. For example, $\{\frac{1}{n}\}$ converges when $X=[0,1]$ and d is the usual Euclidean distance, but does not converge when $X=(0,1]$ or d is the discrete metric. Also, as X and d vary, so will the neighborhoods.

Theorem 6.7. *Let $\{x_n\}$ be a sequence in X, then*

(a) *$\{x_n\}$ converges to x if and only if every neighborhood of x contains x_n for all but finitely many n.*

(b) *If $\{x_n\}$ converges, the limit is unique. That is if $\{x_n\}$ converges to both x and x', then $x=x'$.*

(c) *If $\{x_n\}$ converges, $\{x_n\}$ is bounded.*

Proof. (a) ($\Rightarrow$) By definition, for any $\varepsilon > 0$, there exists N such that for all $n \geq N, d(x_n, x) < \varepsilon$. So finitely many x_n are outside the neighborhood $N_\varepsilon(x)$, for each ε. ($\Leftarrow$) By definition.

(b) Fix an arbitrary $\varepsilon > 0$. As $\{x_n\}$ converges, there exist N and N' such that

$$\begin{aligned} n &\geq N \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}, \\ n &\geq N' \Rightarrow d(x_n, x') < \frac{\varepsilon}{2}. \end{aligned}$$

So when $n \geq \max\{N, N'\}$,

$$d(x, x') \leq d(x, x_n) + d(x_n, x') < \varepsilon.$$

Since ε is arbitrary, $d(x, x') = 0$.

(c) Suppose $x_n \to x$. There exist N such that for each $n \geq N, d(x_n, x) < 1$. Let

$$r = \max\{1, d(x_1, x), d(x_2, x), ..., d(x_N, x)\},$$

then $d(x_n, x) \leq r$, for all n. □

Theorem 6.8. *Fix a sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^k$. Then $\lim_n \mathbf{x}_n = \mathbf{x}$ if and only if $\lim_n x_n^i = x^i$, for each $i \in \{1, ..., k\}$, where $\mathbf{x}_n = (x_n^1, ..., x_n^k)$, and $\mathbf{x} = (x^1, ..., x^k)$.*

Proof. ($\Rightarrow$) Use bounded convergence. Simply note that for each i, $|x_n^i - x^i| \leq |\mathbf{x}_n - \mathbf{x}|$.

($\Leftarrow$) Define $\eta = \frac{\varepsilon}{\sqrt{k}}$. For each i, let $N_i(\eta)$ be such that for each $n \geq N_i(\eta), |x_n^i - x^i| < \eta$. Let $N(\varepsilon) = \max_i\{N_i(\eta)\}$. Then for any $n \geq N(\varepsilon)$,

$$d(x_n, x) = \sqrt{\sum_{i=1}^{k} |x_n^i - x^i|^2} < \sqrt{\sum_{i=1}^{k} (\frac{\varepsilon}{\sqrt{k}})^2} = \varepsilon.$$

□

6.4.2 Cauchy sequences and completeness

Definition 6.7. A sequence $\{x_n\}$ in X is a Cauchy sequence if for every $\varepsilon > 0$ there is an integer N such that $d(x_n, x_m) < \varepsilon$ if $m, n \geq N$.

Definition 6.8. A metric space in which every Cauchy sequence converges is said to be complete.

The difference between convergent sequence and Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter. Every closed subset of any complete metric space is itself a complete metric space. Also, all Euclidean spaces are complete.

Theorem 6.9. *A sequence $\{x_n\}$ in $\mathbb{R}^k$ converges if and only if it is a Cauchy sequence.*

Proof. ($\Leftarrow$) If $\{x_n\}$ is a sequence that converges to x, then there exist N and N' such that for any $m \geq N, n \geq N'$,

$$d(x_m, x) < \frac{\varepsilon}{2}, d(x_n, x) < \frac{\varepsilon}{2}.$$

So for any $m, n \geq \max\{N, N'\}$,

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \varepsilon.$$

This proves that $\{x_n\}$ must be a Cauchy sequence.

($\Rightarrow$) The proof is more involved and is skipped here. $\square$

6.4.3 Upper and lower limits

Definition 6.9. A sequence $\{x_n\}$ of real numbers is said to be monotonically increasing (decreasing) if $x_n \leq (\geq) x_{n+1}$, for each n.

Theorem 6.10. *Suppose $\{x_n\}$ is monotonic. Then $\{x_n\}$ converges if and only if it is bounded.*

Proof. ($\Rightarrow$) Has already been proved in last section.

($\Leftarrow$) Without loss of generality (w.l.o.g.), suppose $\{x_n\}$ is monotonically increasing. Since $\{x_n\}$ is bounded, it must have a least upper bound. Let it be x. Therefore for every

$\varepsilon > 0$, there is an integer N such that $x - \varepsilon < x_N \leq x$. Since $\{x_n\}$ increases, for all $n \geq N$, we have $|x_n - x| < \varepsilon$. That is, $\{x_n\}$ converges to x. $\square$

Definition 6.10. Given a sequence $\{x_n\}$, consider a increasing sequence $\{n_k\}$ of positive integers. Then the sequence $\{x_{n_i}\}$ is called a subsequence of $\{x_n\}$. If $\{x_{n_i}\}$ converges, its limit is called a subsequential limit of $\{x_n\}$.

Definition 6.11. Let $\{x_n\}$ be a sequence of real numbers. Let E be the set of $x \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that $x_{n_i} \to x$ for some subsequence $\{x_{n_i}\}$. The upper and lower limits of $\{x_n\}$ are defined, respectively, as

$$\limsup_{n\to\infty} x_n = \sup E, \liminf_{n\to\infty} x_n = \inf E.$$

For any sequence of real numbers $\{x_n\}$, $\lim_{n\to\infty} x_n = x$ if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.

Example 6.7. Since the set of rational numbers are countable, we can view $\mathbb{Q}$ as a sequence $\{x_n\}$. Then every real number is a subsequential limit, and $\limsup_{n\to\infty} x_n = +\infty, \liminf_{n\to\infty} x_n = -\infty$.

Example 6.8. Let $\{x_n\} = \{1, -1, ..., 1, -1, ...\}$. Then $\limsup_{n\to\infty} x_n = 1, \liminf_{n\to\infty} x_n = -1$.

Equivalently, we can define the upper and low limits as

$$\begin{aligned}
\limsup_{n\to\infty} x_n &= \inf_n \sup_{m\geq n} x_m, \\
\liminf_{n\to\infty} x_n &= \sup_n \inf_{m\geq n} x_m.
\end{aligned}$$

For example, if we let $y_n = \sup_{m\geq n} x_m$, then y_n is a monotonically decreasing sequence. And we know from Theorem 6.10 that as long as y_n is bounded, it will converge.

Theorem 6.11. *If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then*

$$\begin{aligned}
\liminf_{n\to\infty} s_n &\leq \liminf_{n\to\infty} t_n, \\
\limsup_{n\to\infty} s_n &\leq \limsup_{n\to\infty} t_n.
\end{aligned}$$

7 Cardinality and basic topology

This lecture follows mostly chapter 2 in Rudin (1976).

7.1 Finite, coutable and uncountable sets

Definition 7.1. A mapping (function) f from set A to set B is called a $1-1$ mapping if for any $x_1, x_2 \in A$ such that $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$.

Definition 7.2. If $f(A) = B$, we say that the mapping f between A and B is onto.

Definition 7.3. If there exists a $1-1$ onto mapping between A and B, we say that A and B are equivalent (in cardinality), written as $A \sim B$.

Definition 7.4. For any n, let $J_n = \{1, 2, ..., n\}$, and $\mathbb{N} = \{1, 2, ..., n, ...\}$. Then for any set A, we say

(a) A is finite if $A \sim J_n$, for some n.

(b) A is inifinite if it is not finite.

(c) A is countable if $A \sim \mathbb{N}$.

(d) A is uncountable if it is neither finite nor countable.

(e) A is at most countable if it is finite or countable.

Example 7.1. Let $E = (2, 4, 6, ...)$ be the set of even natural numbers. We can show that $E \sim \mathbb{N}$. The $1-1$ onto mapping from $\mathbb{N}$ to E is given by $f(n) = 2n$.

Example 7.2. $\mathbb{Z}$ is countable, i.e., $\mathbb{Z} \sim \mathbb{N}$. This can be proved by showing that the function from $\mathbb{N}$ to $\mathbb{Z}$:

$$f(n) = \begin{cases} \frac{n}{2} \text{ if } n \text{ is even,} \\ -\frac{n-1}{2} \quad \text{if } n \text{ is odd,} \end{cases}$$

is a $1-1$ onto mapping. Only infinite sets can be equivalent to one of its own proper subsets.

Example 7.3. $(-1, 1) \sim \mathbb{R}$. The mapping between $(-1, 1)$ to $\mathbb{R}$ is given by $f(x) = \frac{x}{x^2-1}$. In fact, $(a, b) \sim \mathbb{R}$ for any open interval (a, b).

Theorem 7.1. *The set $\mathbb{Q}$ is countable, and the set $\mathbb{R}$ is uncountable.*

Proof. (i) For each $n \in \mathbb{N}$, let

$$A_n = \{\pm\frac{p}{q} : p + q = n, p, q \in \mathbb{N} \text{ and have no common factor}\}.$$

Then $\mathbb{Q} = \cup_{n\in\mathbb{N}} A_n$. Since each A_n is finite, we can simply count them one by one and form a sequence consists of all rational numbers. Therefore $\mathbb{Q}$ is countable.

(ii) It is sufficient to show that $(0, 1)$ is uncountable. Suppose it is not, then there is a $1 - 1$ onto mapping f between $\mathbb{N}$ and $(0, 1)$, represented as

$$f(n) = 0.a_{n1}a_{n2} \cdots a_{nk} \cdots .$$

We now construct a decimal number $b = 0.b_1b_2 \cdots$, and let $b_n \neq a_{nn}$ for each n. For example, let

$$b_n = \begin{cases} 2 & \text{if } a_{nn} = 1, \\ 1 & \text{if } a_{nn} \neq 1. \end{cases}$$

Since $b \in (0, 1)$, there must be some n, such that $b = f(n)$, that is

$$0.b_1b_2 \cdots = 0.a_{n1}a_{n2} \cdots a_{nn} \cdots ,$$

which implies $b_n = a_{nn}, \forall n$. Contradiction. □

Example 7.4. The union of countably many countable sets and the product of finite countable sets are both countable.

Definition 7.5. Given a set A, the power set of A is the set of all subsets of A, denoted as $P(A)$ or 2^A.

The theorem below states that the power set of any set is always larger than itself. As a result, there is no largest set.

Theorem 7.2 (Cantor's theorem)**.** *Given any set A, there does not exists a function $f : A \to 2^A$ that is onto.*

Proof. We need to show that any function $f : A \to 2^A$ is not onto. For this purpose, it is sufficient to show that there exists $B \in 2^A$ (i.e., $B \subset A$) such that $B \neq f(x)$ for all $x \in A$.

Let

$$B = \{x \in A : x \notin f(x)\}.$$

If $B = f(x_0)$ for any $x_0 \in A$, then

$$x_0 \in f(x_0) \Leftrightarrow x_0 \notin f(x_0).$$

We have a contradiction.

$\square$

Example 7.5 (The Cantor set)**.** Let $C_0 = [0, 1]$. Blow we construct sets C_k inductively by removing the open middle third of each component of C_{k-1}.

$$\begin{aligned}
C_1 &= C_0 \backslash (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], \\
C_2 &= C_1 \backslash \left((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \right) = \left([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \right) \cup \left([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \right), \\
& \cdots
\end{aligned}$$

The Cantor set is defined as

$$C = \cap_{k=0}^{\infty} C_k.$$

This set is an uncountable set as each number in C is uniquely identified by an infinite sequence of binary numbers. However, C has zero length (hence it does not contain any open set). This is because the total length of the open middle intervals removed is

$$\frac{1}{3} + 2(\frac{1}{9}) + 4(\frac{1}{27}) + \cdots + 2^{k-1}(\frac{1}{3^k}) + \cdots = 1.$$

Lastly, we present an application of cardinality on the representation of rational preferences.

Example 7.6 (Lexicographic preference)**.** Suppose $X = [0, 1] \times [0, 1]$, and $(x_1, y_1) \succeq (x_2, y_2)$ if and only if either $x_1 > x_2$ or $x_1 = x_2$ and $y_1 \geq y_2$. The preference relation $\succeq$ is rational (complete and transitive), but cannot be represented by any utility function $u : X \to \mathbb{R}$.

Suppose instead some $u : X \to \mathbb{R}$ represents $\succeq$. For any $x_1, x_2 \in [0,1]$ such that $x_1 > x_2$, since $(x_1, 1) \succ (x_1, 0) \succ (x_2, 1) \succ (x_2, 0)$, $u(x_1, 1) > (x_1, 0) > u(x_2, 1) > u(x_2, 0)$. Therefore, the intervals $\{(u(x,0), u(x,1)) : x \in [0,1]\}$ must be disjoint subsets of $\mathbb{R}$. Since $[0,1]$ is uncountable, there are uncountably many such intervals. However, this is impossible, as each interval contains at least one rational number and rational numbers are countably many.

7.2 Basic topology

Let X be a metric space with a metric $d(\cdot, \cdot)$. All points and sets discussed in this section are elements and subsets of X.

7.2.1 Open and closed sets

Definition 7.6. A point p is an interior point of E if there is a neighborhood N of p such that $N \subset E$. E is open if every point of E is an interior point of E.

Theorem 7.3. *Every neighborhood is an open set.*

Proof. For any $q \in N_r(p)$, let ε be some positive real number such that $\varepsilon < r - d(p,q)$. Then for any $s \in N_\varepsilon(q)$,

$$d(s,p) \leq d(s,q) + d(q,p) < d(p,q) + \varepsilon < r.$$

That is, $s \in N_r(p)$. Proved. $\square$

Definition 7.7. A point p is a limit point of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$. If a point $p \in E$ is not a limit point of E, then p is called an isolated point of E.

Theorem 7.4. *If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.*

Proof. Suppose not, then there is a neighborhood N of p such that N contains only finitely many points, $q_1, .., q_n$, of E. Let

$$r = \min_{1 \leq m \leq n} d(p, q_m).$$

Then the neighborhood $N_r(p)$ contains no point in E. This is in contradiction with the definition of a limit point. $\square$

Definition 7.8. *E is closed if every limit point of E is a point of E.*

That is, a set E is closed if the limit of any convergent sequence in E belongs to E. In other words, E is closed under the operation of taking limit.

Theorem 7.5. *A set E is open if and only if its complement E^c is closed.*

Proof. First, suppose E is open, and x is a limit point of E^c. Since every neighborhood of x intersects with E^c, x is not an interior point of E. So $x \notin E$, that is $x \in E^c$. It follows that E^c is closed.

Secondly, suppose E^c is closed. Pick any $x \in E$. Then $x \notin E^c$ and x is not a limit point of E^c. If every neighborhood of x intersects with E^c, then x would be a limit point of E^c. So there must be some neighborhood N of x, such that $N \cap E^c = \emptyset$, i.e., $N \subset E$. Therefore E is open. $\square$

Example 7.7. Consider the metric space $\mathbb{R}$. Both $\emptyset$ and $\mathbb{R}$ are open and closed. Any interval $(a, b), (a, \infty)$ is open, and any interval $[c, d], [c, \infty)$ is closed.

Example 7.8. $\{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ is neither open nor closed. This set has a limit point which does not belong to it. Similarly, $\mathbb{Q} \subset \mathbb{R}$ is neither open nor closed.

Example 7.9. If X is endowed with the discrete metric, then every $S \subset X$ is both open and closed.

Definition 7.9. *E is dense in X if every point of X is a limit point of E, or a point of E (or both).*

Example 7.10. $\mathbb{Q}$ is dense in $\mathbb{R}$.

Definition 7.10. Let E' be the set of limit points in X, then the closure of E is the set $\bar{E} = E \cup E'$.

The closure of E is the smallest closed set containing E. Finally, we state the following theorem without proof.

Theorem 7.6. *The union of arbitrarily many open sets and the intersection of finitely many open sets are open; the union of finitely many closed sets and the intersection of arbitarily many closed sets are closed.*

7.2.2 Compact sets

The first definition of compact set refers to the concept of open cover. An open cover of a set E in a metric space X is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 7.11. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

That is, a compact set K has the property that, if anybody covers K with an arbitrary collection of open sets, you are always able to (in theory) pick finitely many of those open sets that still covers K. To illustrate this concept, let's see two examples.

Example 7.11. The interval $(0, 1)$ is not compact, because $\{(0, x)\}_{x\in(0,1)}$ is an open cover of it but no finite subcover of $\{(0, x)\}_{x\in(0,1)}$ covers it. Likewise, $\mathbb{R}$ is not compact.

Example 7.12. The interval $[0, 1]$ is compact. Consider any collection of open sets $\{G_\alpha\}$ that covers $[0, 1]$. Let A be the set of all points $x \in [0, 1]$ such that $[0, x]$ can be covered by some finite subcover of $\{G_\alpha\}$. Then $0 \in A$. Let $M = \sup A$.

First, $[0, M]$ can be covered by a finite subcover. This is easy to see if $M = 0$. If $M > 0$, we know that for any $\varepsilon > 0, \exists x \in A$ such that $M - \varepsilon < x < M$. Since $[0, x]$ can be covered by a finite subcover and for small enough ε, $(M - \varepsilon, M]$ is covered by some G_α that contains M, $[0, M]$ is also covered by some finite subcover.

Second, if $M = 1$ then we're done. If $M < 1$, then the finite subcover that covers $[0, M]$ must also cover $[0, M + \delta]$ for small enough δ, because M is covered by one of the open sets. This contradicts with M being the supremum.

An equivalent definition of compactness is through the existence of convergent subsequences.

Theorem 7.7. *A subset K is compact if and only if every sequence in K has a convergent subsequence with limit in K.*

Proof. ($\Rightarrow$) Suppose not. Then there is a sequence $\{x_n\}$ in K which has no limit point in K. That is, either $\{x_n\}$ does not converge at all or it converges but its limit point is not in K. For any $x \in K$, there must be some $\varepsilon_x > 0$ such that the open ball $B_{\varepsilon_x}(x)$ contains no $x_n \neq x$. That is, $B_{\varepsilon_x}(x)$ contains at most one x_n. Observe that $\{B_{\varepsilon_x}(x)\}_{x\in X}$ is an open cover for X. However, since any finite subcover of $\{B_{\varepsilon_x}(x)\}_{x\in X}$ contains at most finitely many x_n, it does not cover $\{x_n\}$; hence it does not cover X. Contradicts.

($\Leftarrow$) we omit this proof. $\square$

While compactness is an abstract concept, for Euclidean spaces, it reduces to the combination of closedness and boundedness of sets. A set S in $\mathbb{R}^k$ is bounded if $\{||\mathbf{x}-\mathbf{y}|| : \mathbf{x}, \mathbf{y} \in S\}$ is bounded.

Theorem 7.8 (Heine-Borel)**.** *A set $S \subset \mathbb{R}^k$ is compact if and only if it is closed and bounded.*

Proof. For notational ease, we prove only for $k = 1$.

($\Rightarrow$) First, if $S \subset \mathbb{R}$ is unbounded, then for every $n \in \mathbb{N}$ there is $x_n \in S$ such that $|x_n| > n$. The sequence $\{x_n\}$ does not converge, so does any of its subsequence. Therefore S cannot be compact. Second, if S is not closed, then there is a convergent sequence $\{x_n\}$ in S whose limit point $x \notin S$. Since any subsequence of $\{x_n\}$ must also converge to x, there is no subsequence which converges at a limit in S. Again, S cannot be compact.

($\Leftarrow$) Suppose there is an open cover $\{G_\alpha\}$ of S which has no finite subcover containing S. Since S is bounded, $S \subset B_r(x_0)$ for some $r \in \mathbb{R}_+$ and $x_0 \in S$. Let $\varepsilon_n = \frac{r}{2^n}$. Note that S can be covered by finitely many open balls with centers in S and radius $\varepsilon_1 = \frac{r}{2}$. Therefore, there must be (at least) one of the open balls which cannot be covered by any finite subcover of $\{G_\alpha\}$. Let it be $B_{\varepsilon_1}(x_1)$. Similarly, the ball $B_{\varepsilon_1}(x_1)$ can be covered by finitely many open balls with radius ε_2, and (at least) one of them, denoted by $B_{\varepsilon_2}(x_2)$, that cannot be covered by any finite subcover of $\{G_\alpha\}$, and satisfies

$$B_{\varepsilon_1}(x_1) \cap B_{\varepsilon_2}(x_2) \neq \emptyset.$$

We can define open balls $B_{\varepsilon_n}(x_n)$ iteratively, and none of them can be covered by finitely many G_α. It can be shown that the centers of these open balls, $x_1, x_2, ..., x_n, ...$, form a Cauchy sequence, hence converge in S. (Try to prove it.) Suppose $\lim_n x_n = x$. Since $\{G_\alpha\}$ is an open cover of S and $x \in S$, there must be some α_0 such that $x \in G_{\alpha_0}$. Also, there must be some open ball $B_\delta(x)$ centered at x such that $B_\delta(x) \subset G_{\alpha_0}$. As $x_n \to x$, for large enough n,

$$B_{\varepsilon_n}(x_n) \subset B_\delta(x) \subset G_{\alpha_0}.$$

That is, G_{α_0} covers $B_{\varepsilon_n}(x_n)$. This contradicts with the fact that $B_{\varepsilon_n}(x_n)$ cannot be covered by finitely many G_α. So S must be compact. $\square$

Example 7.13. The Cantor set is compact, since it is the intersection of bounded and closed sets.

8 Continuity and the Weierstrass theorem

8.1 Continuity

In this section we study continuous functions from a metric space (X, d_X) to a metric space (Y, d_Y).

Definition 8.1. Suppose $f : X \to Y$ and $p \in X$. We say that the limit of function f at p is q, written as $\lim_{x \to p} f(x) = q$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in N_\delta(p)$ implies $f(x) \in N_\varepsilon(q)$.

More generally, we can define the limit of $f : E \to Y$ at p, where $E \subset X$ and p is a limit point of E. Note that p does not need to be in E. To help understand, you can view both X and Y as Euclidean spaces or simply $\mathbb{R}$. From now on, when no confusion may arise, we use $|\cdot|$ to denote all metrics.

Example 8.1. Prove that $\lim_{x \to 2} x^2 = 4$.

The theorem below states an equivalent definition of the limit of a function.

Theorem 8.1. *$\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E, \lim_{n \to \infty} p_n = p$.*

Proof. ($\Rightarrow$) Since $\lim_{x \to p} f(x) = q$, for any $\varepsilon > 0, \exists \delta > 0$, s.t., $\forall x \in E$ with $0 < |x - p| < \delta, |f(x) - q| < \varepsilon$. Suppose $\lim_{n \to \infty} p_n = p$, then $\exists N_0$ s.t. $\forall n \geq N_0, 0 < |p_n - p| < \delta$. For any $\varepsilon > 0$, let $N = N_0$, then for any $n \geq N_0, |f(p_n) - q| < \varepsilon$. That is, $\lim_{n \to \infty} f(p_n) = q$.

($\Leftarrow$) Suppose $\lim_{x \to p} f(x) \neq q$. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E, 0 < |x - p| < \delta$ but $|f(x) - q| > \varepsilon$. Let $\delta_n = \frac{1}{n}$, we can find a corresponding x_n. The sequence $\{x_n\}$ converges to p but $|f(x_n) - q| > \varepsilon, \forall n$. As $\lim_{n \to \infty} f(x_n)$ does not converge to q, we have a contradiction. □

Example 8.2. The Dirichlet function is defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For any $x \in \mathbb{R}$, if $\{x_n\}$ is a sequence of rational numbers that converge to x, then $\lim_{n\to\infty} f(x_n) = 1$. If $\{x_n\}$ is a sequence of irrational numbers that converge to x, then $\lim_{n\to\infty} f(x_n) = 0$. Therefore, this function has no limits at any point.

Definition 8.2. Suppose $f : X \to Y$ and $p \in X$. Then f is continuous at p if $\lim_{x\to p} f(x) = f(p)$. If f is continuous at every x, we say that f is continuous.

By definition, if p is an isolated point of E, that is, if there is no sequence in E that converges to p, then any function on E is continuous at p. This should not bother you as isolated points are far from other points and are rarely interesting.

Theorem 8.2. *A function $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof. ($\Rightarrow$) Suppose f is continuous. Consider any $p \in X$ such that $p \in f^{-1}(V)$. Since V is open, there exists a neighborhood $N_\varepsilon(f(p))$ of $f(p)$ such that $N_\varepsilon(f(p)) \subset V$. As f is continuous, there is a neighborhood $N_\delta(p)$ of p such that for all $x \in N_\delta(p), f(x) \in N_\varepsilon(f(p))$. Therefore $f(x) \in V$ and $x \in f^{-1}(V)$. Since for all $p \in f^{-1}(V)$ there is a neighborhood $N_\delta(p) \subset f^{-1}(V), f^{-1}(V)$ is open.

($\Leftarrow$) Suppose $f^{-1}(V)$ is open for every open set V in Y. Pick any $p \in X$. We know that $N_\varepsilon(f(p))$ is open for any $\varepsilon > 0$. So $f^{-1}(N_\varepsilon(f(p)))$ is open in X. Since $p \in f^{-1}(N_\varepsilon(f(p)))$, it has a open neighborhood $N_\delta(p) \subset f^{-1}(N_\varepsilon(f(p)))$. Obviously, for any $x \in N_\delta(p), f(x) \in N_\varepsilon(f(p))$. Therefore f is continuous. $\square$

Example 8.3. A continuous function may map open sets to non-open sets. Consider the constant function $f : \mathbb{R} \to \mathbb{Z}$ such that $f(x) = 1$ for all $x \in \mathbb{R}$. It maps all open sets to the non-open set $\{1\}$. Also, $f(x) = x^2$ maps $(-1, +1)$ to $[0, 1)$. There are also complicated constructions of examples of functions that maps open sets to open sets but are not continuous.

Example 8.4. Suppose the consumer has a rational preference $\succeq$ on bundles in $\mathbb{R}^K$ and faces a budget set $B(p, w) = \{x \in \mathbb{R}^k : p \cdot x \leq w, x \geq 0\}$. The consumer's preference $\succeq$ is continuous if for any $x, y \in \mathbb{R}^k$ with $x \succ y, \exists \varepsilon > 0$ such that $x' \in N_\varepsilon(x)$ and $y' \in N_\varepsilon(y)$ imply $x' \succ y'$. (Debreu's representation theorem states that if $\succeq$ is continuous, then it can be represented by a continuous utility function $u(\cdot)$.)

The consumer's problem is to find the best bundle according to $\succeq$ from $B(p, w)$. The following result is a direct consequence of Weierstrass theorem. Here we show how it can be proved directly without referring to utility function.

Claim 8.3. *If $\succeq$ is continuous, then the consumer's problem has a solution.*

Proof. For any $x \in B(p, w)$, let $Inferior(x) = \{y \in \mathbb{R}^k : x \succ y\}$. Since $\succeq$ is continuous, every $Inferior(x)$ is an open set in $\mathbb{R}^K$. Suppose instead the consumer's problem has no solution, then every $z \in B(p, w)$ is worse than some $x \in B(p, w)$; that is, for every $z \in B(p, w), \exists x \in B(p, w)$ such that $z \in Inferior(x)$. As a result,

$$B(p, w) \subset \cup_{x \in B(p,w)} Inferior(x).$$

Hence the collection of sets $Inferior(x), x \in B(p, w)$ form an open cover of $B(p, w)$. Since $B(p, w)$ is compact, there must exist finitely many $x_1, \ldots, x_n \in B(p, w)$ such that the $Inferior(x_i)$'s cover $B(p, w)$. That is, for any $z \in B(p, w), z \in Inferior(x_i)$, i.e., $x_i \succ z$, for some x_i. Let $\bar{x}$ be the best bundle among $x_1, \ldots, x_n$. Then $\bar{x} \succ z$ for all $z \in B(p, w)$, a contradiction. □

8.2 The Weierstrass theorem

Let $f : D \subset X \to \mathbb{R}$. We consider conditions for the existence of a solution to the problem

$$\max_{x \in D} f(x).$$

For example, the utility optimization problem subject to compact constraint set. Obviously, if D is finite, the maximum of f is always obtained. The Weierstrass theorem states that we can generate this result to situations when D is compact. This is not quite surprising, as compact sets can be viewed as generalizations of finite sets due to their finite subcover property.

The theorem below states that continuous functions always map compact sets to compact sets.

Theorem 8.4. *Suppose $f : X \to Y$ is continuous. If D is compact in X, then $f(D)$ is compact in Y.*

As a result, if $f : X \to \mathbb{R}$, then $f(D)$ is a compact subset of $\mathbb{R}$, i.e., it is closed and bounded.

Proof. Let $\{V_\alpha\}$ be an open cover of $f(D)$, we will show that $\{V_\alpha\}$ always has a finite subcover. Since f is continuous, for each $V_\alpha, f^{-1}(V_\alpha)$ is open. Obviously, $\cup_\alpha f^{-1}(V_\alpha)$ is an open cover of D. Since D is compact, there are finitely many indices, say $\alpha_1, \ldots, \alpha_n$, such that

$$D \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n}).$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$,

$$f(D) \subset f(f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})) \subset \cup_{\alpha_i} V_{\alpha_i}.$$

Therefore, $\{V_{\alpha_i}\}$ is a subcover of $f(D)$. □

The same result does not hold if we replace compact sets with closed sets.

Example 8.5. Consider $f : D \to \mathbb{R}$ such that $D = [0, \infty)$ and $f(x) = 1/(1 + x^2)$. Then f is continuous and D is closed, but $f(D) = (0, 1]$ is not closed.

Theorem 8.5 (Weierstrass)**.** *Suppose f is a continuous real function on a compact set $D \subset X$, and*

$$M = \sup_{x \in D} f(x), m = \inf_{x \in D} f(x).$$

Then there exist points $p, q \in D$ such that $M = f(p)$ and $m = f(q)$.

That is, on the compact set D, f attains its maximum at p and minimum at q.

Proof. From the previous theorem, since D is compact, $f(D)$ is a compact subset of $\mathbb{R}$. The axiom of completeness states that the supremum and infimum of a bounded set always exist, and since $f(D)$ is also closed, they exist in $f(D)$. □

When D is finite, conditions of the Weierstrass theorem are trivially satisfied. Also, the theorem may fail to hold if either compactness or continuity is relaxed.

Example 8.6. Due to the Weierstrass theorem, the consumer's utility maximization problem $\max u(x)$, s.t. $p \cdot x \leq w, x \geq 0$ has a solution if $u(\cdot)$ is continuous.

9 Correspondences

After solving optimization problems and while we are doing comparative statics, we are usually interested in studying how will the parameters of an optimization problem affect the maximal value and the set of maximizers of the problem. Define, in the utility maximization problem,

$$\begin{aligned} v(p,I) &= \max_{x \in B(p,I)} u(x) \text{ where } B(p,I) = \{x : p \cdot x = I\}, \text{ and} \\ \bar{x}(p,I) &= \arg \max_{x \in B(p,I)} u(x) \equiv \{x \in \mathbb{R}^n : u(x) = v(p,I)\}, \end{aligned}$$

and in the profit maximization problem

$$\begin{aligned} v(p,w) &= \max_{x \in \mathbb{R}^n} pf(x) - w \cdot x, \text{ and} \\ \bar{x}(p,w) &= \arg \max_{x \in \mathbb{R}^n} pf(x) - w \cdot x \equiv \{x \in \mathbb{R}^n : pf(x) - w \cdot x = v(p,w)\}. \end{aligned}$$

The notation "argmax" stands for "the set of arguments in the domain that maximizes", and thus $\arg\max_{x \in D} f(x)$ denotes the set of maximizers $x \in D$ that maximize $f(x)$ over D. The maximal values $v(p,I)$ and $v(p,w)$ map parameters into real numbers, but each of $B(p,I), \bar{x}(p,I)$ and $\bar{x}(p,w)$ maps parameters to sets.

Similarly, the Nash equilibrium solution maps each normal-form game to a set of Nash equilibria. These set-valued (or multi-valued) mappings are called correspondences, as we now formally define.

Definition 9.1. A correspondence from D to Y, written as $\varphi : D \twoheadrightarrow Y$, is a rule that assigns a set $\varphi(x) \subset Y$ to every $x \in D$.

The rightwards two-headed arrow "$\twoheadrightarrow$" is used to distinguish a correspondence from a function (note: there is no standard notation for this). Although correspondences can be defined for general domain and range spaces, since for most of the time we work with only Euclidean spaces, throughout, we restrict attention to $D \subset X = \mathbb{R}^l$ and $Y = \mathbb{R}^k$.

If $\varphi(x)$ contains precisely one element for each x, i.e., if $\varphi(x)$ is always single-valued, then $\varphi(\cdot)$ can be viewed as a function. Similarly, a correspondence $\varphi(\cdot)$ is nonempty, convex, closed or compact valued if $\varphi(x)$ is nonempty, convex, closed or compact, respectively, for every $x \in D$.

9.1 Hemicontinuity

We can also view the correspondence φ as a function that maps D into 2^Y. Given that correspondences are set-valued, the continuity of correspondences is not as direct as that of functions. Given any sequence $\{x_n\}$ such that $x_n \to x$ in D, we now need to consider what it means by saying that a sequence of sets $\varphi(x_n)$ converges to $\varphi(x)$.

Definition 9.2. A correspondence $\varphi : D \subset \mathbb{R}^l \twoheadrightarrow \mathbb{R}^k$ is upper hemicontinuous (uhc) at $x \in D$, if for any open set U such that $\varphi(x) \subset U$, there exists $\varepsilon > 0$ such that

$$\varphi(x') \subset U, \forall x' \in N_\varepsilon(x) \cap D.$$

Definition 9.3. A correspondence $\varphi : D \subset \mathbb{R}^l \twoheadrightarrow \mathbb{R}^k$ is lower hemicontinuous (lhc) at $x \in D$, if for any open set U such that $\varphi(x) \cap U \neq \emptyset$, there exists $\varepsilon > 0$ such that

$$\varphi(x') \cap U \neq \emptyset, \forall x' \in N_\varepsilon(x) \cap D.$$

Definition 9.4. A correspondence is continuous at x if it is both upper and lower hemicontinuous at x.

Let's first discuss the difference between these two concepts and to compare them with the continuity of functions. Let the

$$\begin{aligned} \varphi^{-1}(U) &= \{x \in D : \varphi(x) \subset U\} \text{ and} \\ \varphi^{-1*}(U) &= \{x \in D : \varphi(x) \cap U \neq \emptyset\} \end{aligned}$$

denote the upper and lower inverses of U under φ, which define the set of x at which all values in $\varphi(x)$ belong to U, and the set of x at which some value in $\varphi(x)$ belongs to U, respectively. Recall that a function f is continuous if for any open set U such that $f(x) \in U$, there exists $\delta > 0$ such that $f(x') \in U, \forall x' \in N_\delta(x)$. So we see that both uhc and lhc generalize the continuity of functions to correspondences, but under different views of what it means by "$\varphi(x) \in U$": (i) uhc, all $y \in \varphi(x)$ is in U; (ii) lhc, exists $y \in \varphi(x)$ which is in U. These views lead to different definitions of pre-image.

Theorem 9.1. *Suppose $\varphi : D \twoheadrightarrow \mathbb{R}^k$ is a single-valued correspondence. Then both the uhc and lhc of φ is equivalent to the continuity of φ as a function.*

Remark 9.5. Hemicontinuity can also be defined via the Hausdorff metric on compact subsets of $\mathbb{R}^k$. Recall that $d(y, A) = \inf_{a \in A} d(y, a)$. So we can define $N_\varepsilon(A) \equiv \{y \in \mathbb{R}^k : d(y, A) < \varepsilon\}$. If we restrict attention to compact-valued correspondences, we will have the following equivalent definitions:

We say that $\varphi(x)$ is (metric) upper hemicontinuous at x if for all $\varepsilon > 0, \exists \delta > 0$, such that $x' \in N_\delta(x) \cap D$ implies $\varphi(x') \subset N_\varepsilon(\varphi(x))$, and $\varphi(x)$ is (metric) lower hemicontinuous at x if for all $\varepsilon > 0, \exists \delta > 0$, such that $x' \in N_\delta(x) \cap D$ implies $\varphi(x) \subset N_\varepsilon(\varphi(x'))$.

Remark 9.6. Hemicontinuity is sometimes called semicontinuity, independent of the semicontinuity of functions. More importantly, the definition of hemicontinuity is not fully agreed upon, hence different books and notes may offer somewhat different treatments of it. When reading them, pay attention to: (i) the exact definiton of uhc and lhc; (ii) whether compact-valuedness is implicitly assumed for various results.

Intuitively, if φ is uhc at x, then when we move away from x to some nearby x', there **should not be sudden explosion** in the set of values from $\varphi(x)$ to $\varphi(x')$ (informally, if $x' \in N_\varepsilon(x)$ and $\varphi(x)$ is a subset of any open set U that slightly enlarges $\varphi(x)$, then $\varphi(x')$ should also be a subset of U). If φ is lhc at x, then when we move away from x to some nearby x', there **should not be sudden shrink** in the set of values from $\varphi(x)$ to $\varphi(x')$ (informally, if $x' \in N_\varepsilon(x)$ and $y \in \varphi(x)$, then $\varphi(x')$ should also contain y or something in a small neighborhood U of y). For illustration, let's see the following very simple examples.

Example 9.1. Let $\varphi : [0, 2] \twoheadrightarrow [0, 2]$ be defined as

$$\varphi(x) = \begin{cases} \{1\}, & \text{if } 0 \leq x < 1 \\ [0, 2], & \text{if } 1 \leq x \leq 2. \end{cases}$$

φ is uhc but not lhc at $x = 1$. Note the sudden shrink of $\varphi(x)$ when move away from 1 to its left.

Example 9.2. Let $\varphi : [0, 2] \twoheadrightarrow [0, 2]$ be defined as

$$\varphi(x) = \begin{cases} \{1\}, & \text{if } 0 \leq x \leq 1 \\ [0, 2], & \text{if } 1 < x \leq 2. \end{cases}$$

φ is lhc but not uhc at $x = 1$. Note the sudden explosion of $\varphi(x)$ when move away from 1 to its right.

Example 9.3. Let $\varphi : [0,1] \twoheadrightarrow [0,1]$ be defined as

$$\varphi(x) = \begin{cases} [0,1] \cap \mathbb{Q}, & \text{if } x \in [0,1]\backslash\mathbb{Q}; \\ [0,1]\backslash\mathbb{Q}, & \text{if } x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

φ is not uhc (at $x \in [0,1] \cap \mathbb{Q}$) but is lhc. In particular, for any $x \in [0,1] \cap \mathbb{Q}, \varphi(x) \subset (0,1)$, but for any irrational number x' nearby, $\varphi(x')$ contains 0 and 1, which are not included in $(0,1)$.

The theorem below presents equivalent definitions of uhc and lhc via sequences.

Theorem 9.2. *Let $\varphi : D \subset \mathbb{R}^l \twoheadrightarrow \mathbb{R}^k$ be a correspondence.*

1. *Suppose φ is nonempty and compact-valued. Then φ is upper hemicontinuous at $x \in D$ if and only if for every sequence $x_n \to x$ and $y_n \in \varphi(x_n)$, there is a convergent subsequence of $\{y_n\}$ that converges to some $y \in \varphi(x)$.*

2. *φ is lower hemicontinuous at x if and only if $x_n \to x$ and $y \in \varphi(x)$ imply that there is a sequence $y_n \in \varphi(x_n)$ with $y_n \to y$.*

Proof. (1, $\Rightarrow$) Suppose $\varphi(x)$ is uhc at $x, x_n \to x$ and $y_n \in \varphi(x_n)$. Since $\varphi(x)$ is compact, there is a bounded open set U such that $\varphi(x) \subset U$. The uhc of φ implies that there exists $\varepsilon > 0$, such that $\forall x' \in N_\varepsilon(x), \varphi(x') \subset U$. Hence there exists N such that $\forall n \geq N, y_n \in \varphi(x_n) \subset U$. Therefore, $\{y_n\}$ is eventually in U, hence bounded, and has a convergent subsequence. The limit y of this convergent subsequence has to be in $\varphi(x)$. Otherwise, let $U' = \{y' : d(y', \varphi(x)) < \frac{1}{2}d(y, \varphi(x))\}$. Then $\varphi(x) \subset U'$ but $y \notin \bar{U}'$, where $\bar{U}'$ is the closure of U'. Then again, y_n is eventually in U' and cannot have any subsequence converge to y, which is outside of $\bar{U}'$.

(1, $\Leftarrow$) Suppose for every sequence $x_n \to x$ and $y_n \to \varphi(x_n)$, there is a convergent subsequence of $\{y_n\}$ that converges to some $y \in \varphi(x)$. Suppose instead $\varphi(x)$ is not uhc. Then there is an open set U with $\varphi(x) \subset U$, and a sequence (let $\varepsilon_n = \frac{1}{n}$, for example) $z_n \to x, y_n \in \varphi(z_n)$, but $y_n \notin U$. Since $U \supset \varphi(x)$ is open, such a sequence $\{y_n\}$ cannot have any subsequence that converges into $\varphi(x)$. We have a contradiction.

(2, $\Rightarrow$) Suppose $\varphi(x)$ is lhc, $x_n \to x$ and $y \in \varphi(x)$. For each natural number p, the set $G_p = \{y' \in \mathbb{R}^k : d(y', y) < \frac{1}{p}\}$ is open, and $G_p \cap \varphi(x) \neq \emptyset$. Since $\varphi(x)$ is lhc and $x_n \to x$, there exists N_p such that for all $n \geq N_p, \varphi(x_n) \cap G_p \neq \emptyset$. Without loss of generality, assume $N_p < N_{p+1}, \forall p$. For any $N_p \leq n < N_{p+1}$, pick y_n from $\varphi(x_n) \cap G_p$, and for any $n < N_1$, pick y_n arbitrarily from $\varphi(x_n)$. Then for any $n \geq N_p, d(y_n, y) < \frac{1}{p}$; that is, $y_n \to y$.

(2, $\Leftarrow$) Suppose whenever $x_n \to x$ and $y \in \varphi(x)$, there is a sequence $y_n \in \varphi(x_n)$ with $y_n \to y$. Let U be any open set such that $\varphi(x) \cap U \neq \emptyset$, and let $y \in \varphi(x) \cap U$. Suppose instead φ is not lhc. Then for each n, there is $x_n \in N_{\frac{1}{n}}(x)$ with $\varphi(x_n) \cap U = \emptyset$. Since $x_n \to x$, by assumption, there exists $y_n \in \varphi(x_n)$ such that $y_n \to y$. But this is not possible, since all such $\{y_n\}$ are outside of an open set U, which contains y. We have a contradiction.

□

Definition 9.7. The graph of a correspondence $\varphi : D \subset \mathbb{R}^l \twoheadrightarrow \mathbb{R}^k$ is defined as

$$Gr(\varphi) = \{(x, y) \in D \times \mathbb{R}^k : y \in \varphi(x)\}.$$

When $Gr(\varphi)$ is closed, we say that φ has closed graph. Obviously, if φ has closed graph, then $\varphi(x)$ is closed in $\mathbb{R}^k$ for all $x \in D$; that is, φ is closed-valued.

Definition 9.8. A correspondence $\varphi : D \subset \mathbb{R}^l \twoheadrightarrow \mathbb{R}^k$ is locally bounded if for every $x \in D$, there exists $\varepsilon > 0$ and a bounded set $Y(x) \subset \mathbb{R}^k$, such that for all $x' \in N_\varepsilon(x) \cap D, \varphi(x') \subset Y(x)$.

Theorem 9.3. *Suppose φ is compact-valued. Then*

1. *if φ is uhc, then it has closed graph, and*
2. *if φ is also locally bounded, then it has closed graph implies it is uhc.*

Proof. (1) Suppose φ is uhc and $(x_n, y_n) \to (x, y)$. We only need to show that $(x, y) \in Gr(\varphi)$; that is, $y \in \varphi(x)$. Since φ is uhc and compact-valued, from Theorem 9.2, there is a convergent subsequence of y_n that converges to some $y' \in \varphi(x)$. Since $y_n \to y$, it has to be that $y = y' \in \varphi(x)$.

(2) Suppose φ is locally bounded and has closed graph, but it is not uhc. Then there is an open set U such that $\varphi(x) \subset U$, but there is no $N_\varepsilon(x) \subset \varphi^{-1}(U)$. In particular, for each n, there exists $x_n \in N_{\frac{1}{n}}(x)$ such that $\varphi(x_n) \cap U^c \neq \emptyset$. That is, there exists $x_n \to x$ and $y_n \in \varphi(x_n)$ such that $y_n \notin U$. Since φ is locally bounded, there exist a bounded set

$Y(x)$ and N, such that for all $n \geq N, y_n \in \varphi(x_n) \subset Y(x)$. Therefore, y_n has a convergent subsequence y_{n_p}. Since φ has closed graph, $x_{n_p} \to x$ and $y_{n_p} \to y$, we must have $y \in \varphi(x)$. However, this would contradict with $y_n \notin U \supset \varphi(x), \forall n$ and U is open.

□

Example 9.4. Let $\varphi : [0, \infty) \twoheadrightarrow \mathbb{R}$ be a correspondence such that, $\varphi(x) = \{\frac{1}{x}\}$ if $x > 0$ and $\varphi(0) = \{0\}$. This correspondence is equivalent to a function. It is not locally bounded because at $x = 0$, for no $\varepsilon > 0$ and can we find such a bounded $Y(0) \subset \mathbb{R}$.

Moreover, it is compact-valued and has closed graph, but is not uhc, as it is not continuous when viewed as a function.

9.2 The Maximum Theorem

Consider a real-valued function $f(\cdot, \theta) : S \subset \mathbb{R}^n \to \mathbb{R}$, where $\theta \in \Theta \subset \mathbb{R}^k$ is the parameter of this function, and Θ is the set of parameters. Suppose S is compact and let $C : \Theta \twoheadrightarrow S$ be the constraint correspondence. Then

$$\max_{x \in C(\theta)} f(x, \theta)$$

is a parameterized constrained optimization problem. Define the value and the set of maximizers of the optimization problem, respectively, as

$$\begin{aligned} v(\theta) &= \max_{x \in C(\theta)} f(x, \theta), \text{ and} \\ \bar{x}(\theta) &= \arg\max_{x \in C(\theta)} f(x, \theta). \end{aligned}$$

Example 9.5. In the standard utility maximization problem, $\theta = (p, I), C(\theta) = \{x : p \cdot x = I\}, f(x, \theta) = u(x)$ for all θ, and $v(p, I)$ (the indirect utility function) and $\bar{x}(p, I)$ (the Marshallian demand function) were previously defined.

The following theorem is very important in mathematic economics. It is often used to prove conditions needed to apply the Brouwer and Kakutani fixed-point theorems.

Theorem 9.4 (Berge's Maximum Theorem)**.** *Suppose $f : S \times \Theta \to \mathbb{R}$ is a continuous function and $C : \Theta \twoheadrightarrow 2^S$ is a compact-valued continuous correspondence (i.e., both uhc and lhc in θ). Then*

1. *$v(\theta)$ is a continuous function of θ, and*

2. *$\bar{x}(\theta)$ is a compact-valued upper hemicontinuous correspondence of θ.*

Proof. Due to the Weierstrass theorem and the compact-valuedness of $C(\theta)$, for each θ, there exists a maximizer for each parameterized optimization problem. Hence $\bar{x}(\theta)$ is nonempty for each θ. For any $\theta_n \to \theta$ and any $\bar{x}_n \in \bar{x}(\theta_n) \subset C(\theta_n)$ (so $f(\bar{x}_n, \theta_n) = v(\theta_n)$), since $C(\cdot)$ is compact-valued and uhc, by Theorem 9.2, there is a convergent subsequence $\{\bar{x}_{n_k}\}$ such that $\bar{x}_{n_k} \to x$, for some $x \in C(\theta)$.

Now we show that $x \in \bar{x}(\theta)$. Suppose not, then there exists $x' \in \bar{x}(\theta)$ such that $f(x', \theta) > f(x, \theta)$. Since $C(\cdot)$ is lhc, $\theta_{n_k} \to \theta$, and $x' \in C(\theta)$, there is a sequence $x_{n_k} \in C(\theta_{n_k})$, such that $x_{n_k} \to x'$. We have constructed by now,

$$(\bar{x}_{n_k}, \theta_{n_k}) \to (x, \theta) \text{ and } (x_{n_k}, \theta_{n_k}) \to (x', \theta).$$

By definition, $\bar{x}_{n_k} \in \bar{x}(\theta_{n_k})$, so $f(\bar{x}_{n_k}, \theta_{n_k}) \geq f(x_{n_k}, \theta_{n_k})$. Therefore, due to the continuity of f,

$$f(x, \theta) = \lim_k f(\bar{x}_{n_k}, \theta_{n_k}) \geq \lim_k f(x_{n_k}, \theta_{n_k}) = f(x', \theta).$$

This contradicts with $f(x', \theta) > f(x, \theta)$. Results of the theorem are now immediate.

1. Suppose $v(\cdot)$ is not continuous at θ. Then there exist $\varepsilon > 0$ and $\theta_n \to \theta$ such that for all n, $|v(\theta_n) - v(\theta)| \geq \varepsilon$. From the above, this is not possible, because we can find a subsequence $\theta_{n_k} \to \theta$ such that $v(\theta_{n_k}) = f(\bar{x}_{n_k}, \theta_{n_k}) \to f(x, \theta) = v(\theta)$.

2. We have shown that for any $\theta_n \to \theta$ and $\bar{x}_n \in \bar{x}(\theta_n)$, there is a convergent subsequence $\bar{x}_{n_k}$ that converges to some $x \in \bar{x}(\theta)$. By Theorem 9.2, $\bar{x}(\theta)$ is uhc in θ. The compactness of $\bar{x}(\theta)$ is straightforward from the compactness of $C(\theta)$ and the continuity of f.

$\square$

Example 9.6. When $f(x, \theta)$ is strictly quasiconcave in x and continuous, and $C(\cdot)$ is convex and compact-valued, then for each $\theta, \bar{x}(\theta)$ is single-valued. By Berge's maximum theorem, $\bar{x}(\theta)$ is continuous as a function.

From Berge's theorem, the continuity of f and C cannot ensure the lsc of $\bar{x}(\theta)$; that is, part of the continuity is lost during optimization. This can be illustrated by the following example.

Example 9.7. Let $S = \Theta = C(\theta) = [0, 1]$, for all θ, and consider $f : S \times \Theta \rightarrow [0, 1]$ defined by $f(x, \theta) = \theta x$. Obviously, $\bar{x}(\theta) = \{1\}$, for all $\theta > 0$ and $\bar{x}(0) = [0, 1]$. So $\bar{x}(\theta)$ is not lhc at 0.

10 Dynamic programming

10.1 Contraction mapping (fixed-point) theorem

Definition 10.1. Let (X, d) be a metric space. A mapping $f : X \to X$ is a contraction mapping if there exists a number $\beta \in (0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \beta d(x, y).$$

It is easy to prove that every contraction mapping is continuous. Function f moves (maps) each point $x \in X$ to some point $f(x) \in X$. If x stays at where it is when being moved, then x is said to be a fixed point of f.

Definition 10.2. Suppose $f : X \to X$. If $f(x) = x$, then we say that x is a fixed point of f.

Example 10.1. The function $f(x) = \frac{x}{2}$ is a contration mapping from $\mathbb{R}$ to $\mathbb{R}$, and it has a fixed point $x = 0$.

Theorem 10.1 (Contraction mapping)**.** *If X is a complete metric space and $f : X \to X$ is a contraction mapping, then there exists one and only one $x \in X$ such that $f(x) = x$.*

Proof. Pick $x_0 \in X$ arbitrarily, and define $\{x_n\}$ recursively by setting

$$x_{n+1} = f(x_n), \text{ for } n = 0, 1, 2, \ldots.$$

For $n \geq 1$, we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \beta d(x_n, x_{n-1}).$$

Hence

$$d(x_{n+1}, x_n) \leq \beta^n d(x_1, x_0).$$

And for $n < m$,

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \\
&\leq (\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}) d(x_1, x_0) \\
&= \beta^n (1 + \beta + \cdots + \beta^{m-n-1}) d(x_1, x_0) \\
&\leq \beta^n \frac{1 - \beta^{m-n}}{1 - \beta} d(x_1, x_0) \\
&\leq \beta^n \frac{1}{1 - \beta} d(x_1, x_0).
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, we can find N such that $m, n \geq N$ implies $d(x_n, x_m) < \varepsilon$. That is, $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ converges to some $x \in X$.

Since f is a contraction mapping, it is continuous. Therefore

$$f(x) = \lim_n f(x_n) = \lim_n x_{n+1} = x.$$

That is, x is a fixed point of f. To see the uniqueness of the fixed point, observe that if $f(x) = x$ and $f(y) = y$, then $d(f(x), f(y)) = d(x, y) \leq \beta d(x, y)$. Since $\beta < 1$, we must have $d(x, y) = 0$ and therefore $x = y$. $\square$

This contraction mapping theorem is also called the Banach fixed point theorem. Note that while whether a function is a contraction mapping depends on the metric that is being used (under the discrete metric no function is a contraction mapping), the concept of fixed point does not. Therefore, to apply the theorem, we first need to find an appropriate metric. This theorem has many applications, among which the Bellman equation is of particular interest to economists.

10.2 The optimal growth model

Suppose an agent is endowed with a captital stock $k_0 > 0$, her utility function is $u(c)$, and she can use k units of captital to produce an output $f(k)$. In each period $t \in \{0, 1, \ldots\}$, given her current capital stock k_t, the agent produces $f(k_t)$, saves k_{t+1} as the capital stock of the next period, and consumes $c_t = f(k_t) - k_{t+1}$. The agent discounts next period's utility by $\beta \in (0,1)$.

We assume that both $u(\cdot)$ and $f(\cdot)$ are countinuously differentiable, strictly concave and strictly increasing on $\mathbb{R}_+$. Moreover, assume $u(\cdot)$ is bounded, $u'(0) = f'(0) = \infty$ (Inada's condition), $f'(\infty) = 0$, and $f(0) = 0$.

At $t = 0$, the agent chooses an saving (or consumption) path that maximizes her total discounted utility. The agent's problem is a infinite-horizon dynamic optimization problem

$$\begin{aligned}
&\max_{\{k_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}),\\
&\quad 0 \le k_{t+1} \le f(k_t), \forall t = 0, 1, \ldots\\
&\quad k_0 > 0 \text{ is given.}
\end{aligned}$$

In an equivalent manner, we can define $\{c_t\}_{t=0}^{\infty}$ as the choice variables, so that the agent decides how much to consume in each period. Note that since u is bounded, the value of this problem is bounded.

Example 10.2 (Cake-eating)**.** The problem can be conveniently viewed as a cake-eating problem. The agent has a cake of size k_0 and decides in each period how much to eat and how much to save for future. If $f(k) = k$, then each unit of cake saved for future values a unit. If $f(k) = (1-\delta)k$, it means that the saved cake depreciates by δ each period.

10.3 Finite-horizon problem: direct solution

Before solving the infinite-horizon problem, let's first try to gain some intuition from solving the much simpler finite version of it. Suppose the agent lives only until period T, and T is known ex ante. The agent chooses $(k_1, \ldots, k_{T+1}) \in \mathbb{R}^{T+1}$ to maximize her total discounted utility at $t = 0$,

$$\begin{aligned}
&\max_{\{k_t\}_{t=1}^{T+1}} \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}),\\
&\text{s.t. } 0 \le k_{t+1} \le f(k_t), \forall 0 \le t \le T\\
&k_0 > 0 \text{ is given.}
\end{aligned}$$

We can now set up the Lagrangean of this problem and solve it using Kuhn-Tucker conditions.

$$L(k_1, \ldots, k_{T+1}, \lambda) = \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}) + \sum_{t=0}^{T} \lambda_t [f(k_t) - k_{t+1}] + \sum_{t=1}^{T+1} \lambda'_t k_t.$$

Although this looks to be complicated, our assumptions on $u(\cdot)$ and $f(\cdot)$ have basically ruled out all corner solutions. Note that constraint set is a convex subset of $\mathbb{R}^{T+1}$, and since $u'(0) = f'(0) = \infty$, at the optimum, $0 < \bar{k}_t < f(\bar{k}_{t+1}), \forall t = 0, \ldots, T$, and obviously $\bar{k}_{T+1} = 0$. So except for $t = T+1$, we only need to care about first-order conditions.

The first-order conditions (FOC) are

$$u'(f(k_{t-1}) - k_t) = \beta u'(f(k_t) - k_{t+1}) \cdot f'(k_t), \forall t = 1, \ldots, T.$$

Different from the usual FOC for a static problem, here we have a sequence of FOCs, one for each period t. These intertemporal FOCs are called the **Euler equations** in dynamic programming; they are necessary for optimal resource allocation across time. In terms of optimal consumption, these equations can be equivalently expressed as

$$u'(c_{t-1}) = \beta u'(c_t) \cdot f'(k_t), \forall t = 1, \ldots, T.$$

Let $v(k_0)$ be the value of the optimization problem. Due to the Envelope theorem,

$$\begin{aligned} v'(k_0) &= u'(f(k_0) - k_1) \cdot f'(k_0) \\ &= \beta^t u'(f(k_t) - k_{t+1}) \cdot f'(k_0) f'(k_1) \cdots f'(k_t), \forall t = 1, \ldots, T. \end{aligned}$$

To see the intuition of the FOCs, consider saving one more unit of capital from period $t-1$ to period t (or any other later period) on the optimal path. By doing so, the marginal decrease of utility in period $t-1$ is $u'(c_{t-1})$ and the marginal increase of production output in period t is $f'(k_t)$. If this amount is consumed in period t, the total increase of period t utility is $f'(k_t) \cdot \beta u(c_t)$ (discounted to period $t-1$ for direct comparison), and if it is consumed in period $t+1$, the total increase of utility is $f'(k_t) f'(k_{t+1}) \beta^2 u'(c_{t+1})$. On the optimal path, due to the FOCs, the loss and potential benefit at any later period are equal, hence no such reallocation of resources across time is benefitable. As a result, on the optimal path, if the agent is given an extra unit of capital, she will be indifferent among consuming it in any period.

The FOCs for $t = 1, \ldots, T$ form a second-order difference equation with two terminal conditions: k_0 is given, and $k_{T+1} = 0$. In general, it is solvable.

Example 10.3. Suppose $u(c) = \ln c$ and $f(k) = k$. Then the FOCs are $k_t - k_{t+1} =$

$\beta(k_{t-1} - k_t), \forall t = 1, \ldots, T$. Therefore,

$$\begin{aligned} k_0 - k_{T+1} &= (k_0 - k_1) + (k_1 - k_2) + \cdots + (k_T - k_{T+1}) \\ &= (1 + \beta + \cdots + \beta^T)(k_0 - k_1). \end{aligned}$$

Since $k_{T+1} = 0$, we have

$$c_0 = k_0 - k_1 = \frac{1}{1 + \beta + \cdots + \beta^T} k_0.$$

And $k_2, \ldots, k_T$ can be derived inductively.

10.4 Finite-horizon dynamic programming

Suppose we have already known that for each finite number T and initial endowment k_0, the value of the T-period optimal growth problem is $v_T(k_0)$. Then by **backwards induction** or the **Principle of optimality** (due to Richard Bellman), which breaks a problem into smaller subproblems, we have the **Bellman equation**

$$v_{T+1}(k_0) = \max_{k_1} u(f(k_0) - k_1) + \beta v_T(k_1).$$

Given that the agent faces a $T+1$ period problem today, if she knows the optimal value $v_T(k_1)$ of a T-period problem under each endowment k_1, she only needs to trade-off between today's consumption and the T-period problem which she will face tomorrow. The agent anticipates that given each of today's consumption $f(k_0) - k_1$, she will always be maximizing in the future, so the consequence on future is $\beta v_T(k_1)$. By taking the consequences of different k_1's into consideration, she choose the optimal k_1.

From the first-order condition of this problem and the Envelope theorem,

$$\begin{aligned} \beta v_T'(k_1) &= u'(f(k_0) - k_1), \text{ and} \\ v_{T+1}'(k_0) &= u'(f(k_0) - k_1) f'(k_0) = \beta v_T'(k_1) f'(k_0). \end{aligned}$$

By now it seems that we're going nowhere, since we don't actually know $v_T(k_0)$ for each T and k_0. This is not quite true. We know $v_1(k_0)$ for each k_0, since $T = 1$ implies the agent only lives for one period, i.e., $v_1(\cdot) = u(\cdot)$.

The value of the $T = 2$ period problem, given endowment k_0, is

$$\begin{aligned} v_2(k_0) &= \max_{k_1} u(f(k_0) - k_1) + \beta v_1(k_1) \\ &= \max_{k_1} u(f(k_0) - k_1) + \beta u(k_1). \end{aligned}$$

We can then solve for k_1 through the FOC

$$u'(f(k_0) - k_1) = \beta u'(k_1).$$

By plugging the solution of k_1 as a function of k_0 back into $v_2(k_0)$, we will have the functional form of $v_2(k_0)$. We can then derive the functional form $v_T(k_0)$ for each $T \geq 3$ recursively.

Example 10.4. Suppose $u(c) = \ln c$ and $f(k) = k$. Now consider $v_T(k_0)$. Obviously, $v_1(k) = u(k) = \ln k, \forall k$. For $T = 2$, due to the FOC,

$$\frac{1}{k_0 - k_1} = \beta \frac{1}{k_1} \Rightarrow \bar{k}_1 = \frac{\beta}{1+\beta} k_0.$$

Hence the functional form for $T = 2$ is $v_2(k_0) = \ln(k_0 - \bar{k}_1) + \beta \ln \bar{k}_1 = A_2 + B_2 \ln k_0$, where $A_2 = \ln \frac{1}{1+\beta} + \beta \ln \frac{\beta}{1+\beta}, B_2 = 1 + \beta$. For $T \geq 3, v_T(k_0)$ can be derived recursively. The value functions turn out to have the same form $v_T(k_0) = A_T + B_T \ln k_0$.

10.5 Infinite-horizon dynamic programming

We now come back to the infinite period dynamic optimization problem.

$$\begin{aligned} &\max_{\{k_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}), \\ &\text{s.t. } 0 \leq k_{t+1} \leq f(k_t), \forall t = 1, 2, \ldots \\ &k_0 = k > 0 \text{ is given.} \end{aligned}$$

Let $v(\cdot)$ be the value function of this optimization problem. Then from the principle of optimality, we have the following **Bellman equation**,

$$v(k) = \max_{0 \le k' \le f(k)} u(f(k) - k') + \beta v(k'),$$

where k is the initial endowment and k' is the capital saved for tomorrow. As the endowment of tomorrow's problem, k' will generate an optimal value of $v(k')$, according to the value function $v(\cdot)$. Note that now time does not enter this functional equation. This is the essence of the **stationarity** of the problem. Since the agent faces an infinite-horizon problem today, after she decides how much to consume/save today, the problem left for tomorrow is still an infinite-horizon problem. The difference between today and tomorrow's problem lies only in the endowments; consequently, the functional form of the value function does not change.

Theorem 10.2. *The solution $v(\cdot)$ to the Bellman equation is the value function of the infinite-horizon dynamic optimization problem.*

Proof. First, let $k_1, k_2, \ldots$ be any feasible saving path. Then

$$\begin{aligned} v(k_0) &= \max_{0 \le k' \le f(k_0)} u(f(k_0) - k') + \beta v(k') \\ &\ge u(f(k_0) - k_1) + \beta v(k_1) \\ &\ge u(f(k_0) - k_1) + \beta[u(f(k_1) - k_2) + \beta v(k_2)] \\ &\ge \cdots \\ &\ge \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}). \end{aligned}$$

To see the first inequality, note that since k_1 is the prescribed saving level, it cannot perform better than the optimal one which generates $v(k_0)$. All other inequalities follow the same intuition.

Second, given $v(\cdot)$, let $\bar{k}_1 \in \arg\max_{0 \le k' \le f(k_0)} u(f(k_0) - k') + \beta v(k')$. And recursively, let $\bar{k}_2 \in \arg\max_{0 \le k' \le f(\bar{k}_1)} u(f(\bar{k}_1) - k') + \beta v(k')$, and $\bar{k}_t \in \arg\max u(f(\bar{k}_{t-1}) - k') + \beta v(k'), \forall t > 2$.

All together, we have

$$\begin{aligned}
v(k_0) &= u(f(k_0) - \bar{k}_1) + \beta v(\bar{k}_1) \\
&= u(f(k_0) - \bar{k}_1) + \beta[u(f(\bar{k}_1) - \bar{k}_2) + \beta v(\bar{k}_2)] \\
&= \cdots \\
&= \sum_{t=0}^{\infty} \beta^t u(f(\bar{k}_t) - \bar{k}_{t+1}).
\end{aligned}$$

Therefore, there actually exists a saving path $\bar{k}_1, \bar{k}_2, \ldots$ that generates value $v(k_0)$. As a result, $v(k_0)$ is the (optimal) value of the dynamic optimization problem.

$\square$

Theorem 10.3. *There exists a unique solution $v(\cdot)$ to the Bellman equation.*

Assume that the endowment k belongs to K, which is a compact subset of $\mathbb{R}$. Recall that $u(f(k) - k')$ is continuous in (k, k'). Let $C(K) = \{v : K \to \mathbb{R}\}$ be the set of real-valued continuous functions on K, endowed with the sup-norm. So for functions $v, w \in C(K)$, $||v - w|| = \sup_{k \in K} |v(k) - w(k)|$.

Proof. For each function $v \in C(K)$, define

$$Tv(k) = \max_{0 \le k' \le f(k)} u(f(k) - k') + \beta v(k').$$

Note that both sides of this equation are functions of k. Since $u(f(k) - k') + \beta v(k')$ is continuous in (k, k'), and the constraint correspondence that maps k to $[0, f(k)]$ is continuous, due to Berge's Maximum theorem, $Tv(k)$ is continous. That is, $Tv \in C(K)$; T maps from $C(K)$ to $C(K)$.

Let $\bar{k}$ be the maximizer of right-hand-side optimization problem under function v. Then for each $k \in K$ and function $w \in C(K)$,

$$\begin{aligned}
Tv(k) &= u(f(k) - \bar{k}) + \beta v(\bar{k}) \\
&= u(f(k) - \bar{k}) + \beta w(\bar{k}) + \beta v(\bar{k}) - \beta w(\bar{k}) \\
&\le Tw(k) + \beta ||v - w||.
\end{aligned}$$

Similarly, for each $k \in K, Tw(k) \leq Tv(k) + \beta||v - w||$. Together we have,

$$||Tv - Tw|| = \sup_{k \in K} |Tv(k) - Tw(k)| \leq \beta||v - w||.$$

Since $\beta \in (0, 1)$, T is a contraction mapping on $C(K)$. Due to the Contraction mapping theorem, T has a unique fixed point v^* such that $Tv^* = v^*$. That is, v^* solves the Bellman equation.

$\square$

From the FOC of the Bellman equation and the Envelope theorem, we have

$$\begin{aligned} \beta v'(k') &= u'(f(k) - k'), \text{ and} \\ v'(k) &= u'(f(k) - k') \cdot f'(k). \end{aligned}$$

Let k'' be the optimal saving of tomorrow's problem, then $v'(k') = u'(f(k') - k'') \cdot f'(k')$. Together we have the **Euler equation**

$$u'(f(k) - k') = \beta u'(f(k') - k'') \cdot f'(k').$$

In terms of today and tomorrow's optimal consumption, $u'(c) = \beta u'(c') \cdot f'(k')$. And along the optimal saving/consumption path, this condition holds in every period.

In our setup, k is the state variable which describes the current endowment, and k' is the control variable, which is the choice to be made. The function $k' = g(k)$ that maps k to k' is called the policy function. If the value function $v(k)$ is given, we can derive $k' = g(k)$ from it. In applications, instead, if we observe states and the respective actions taken, we can use those data to estimate parameters of the policy function.

However, it is in general not possible to find a closed form solution to the value function. Researchers usually try to characterize some properties of the solution, or if necessary, solve it numerically. The direct way to do that is to start with a guess of a feasible policy function $g_0(k)$ (for example, $g_0(k) = \theta f(k), \theta \in (0, 1)$). This policy function will generate a value function $v_0 \in C(K)$. By applying the contraction mapping T, we can generate a new value function

$$v_1(k) = Tv_0(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta v_0(k'),$$

and recursively, a sequence of value functions $v_2 = Tv_1, v_3 = Tv_2, \ldots$. According to the contraction mapping theorem, Tv_n will eventually converge (under the sup-norm) to the value function v–the fixed point of T. Since $v_0(k) = u(f(k) - g_0(k)) + \beta v_0(g_0(k))$, by definition, $v_1(k) \geq v_0(k), \forall k$. Due to the same argument, each time we apply T, the value function improves.

That said, in rare cases, we may be able to guess the functional form of the value function, based on the functional forms of finite-horizon problems.

Example 10.5. Suppose $u(c) = \ln c$ and $f(k) = k$. From Example 10.4, a good guess of the solution to the infinite-horizon problem takes the form $v(k) = A + B \ln k$. If so, the Bellman equation is,

$$A + B \ln k = \max_{k'} \ln(k - k') + \beta(A + B \ln k'), \text{ for all } k.$$

The FOC on k' is $-\frac{1}{k-k'} + \beta B \frac{1}{k'} = 0$. Solving for k', we have $k' = \frac{\beta B}{1+\beta B} k$.

Plugging this into the Bellman equation,

$$\begin{aligned} A + B \ln k &= \ln \frac{1}{1 + \beta B} k + \beta(A + B \ln \frac{\beta B}{1 + \beta B} k) \\ &= [\beta A + \beta B \ln \beta B - (1 + \beta B) \ln(1 + \beta B)] + (1 + \beta B) \ln k. \end{aligned}$$

Since this equation holds for all $k, B = 1 + \beta B$ and $B = \frac{1}{1-\beta}$. Similarly, $A = \frac{1}{(1-\beta)^2}[\beta \ln \beta + (1 - \beta) \ln(1 - \beta)]$.

The policy function is

$$\begin{aligned} k' &= g(k) = \frac{\beta B}{1 + \beta B} k = \beta k, \text{ and} \\ c &= k - k' = (1 - \beta)k. \end{aligned}$$

11 Fixed-point theorems

Fixed points are closely related to the concepts of stability and equilibrium. This is because by definition, when the process described by f reaches a fixed point, it stays there forever. Fixed point theorems provide conditions for the existence of fixed points and are very useful tools in economics. For different applications, we may need different such theorems. We've studied the contraction mapping theorem; in this lecture, we study more of the commonly used fixed point theorems.

11.1 Brouwer's fixed point theorem

Definition 11.1. For a function $f : A \to A$, if $x = f(x)$ for some $x \in A$, then x is a fixed point of f.

Theorem 11.1 (Brouwer)**.** *Suppose $A \subset \mathbb{R}^n$ is nonempty, compact and convex, and $f : A \to A$ is a continuous function. Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x = f(x)$.*

The proof of this theorem is very complicated, and we won't go as far to prove it. However, the proof to the simplest version of the theorem, where $A = [0, 1]$, is pretty straightforward.

Example 11.1. Suppose $f : [0, 1] \to [0, 1]$ is continuous. Define $\phi(x) = f(x) - x$. Then $\phi(0) = f(0) - 0 \geq 0, \phi(1) = f(1) - 1 \leq 0$. By the continuity of ϕ and the intermediate value theorem, there exists $x \in [0, 1]$ such that $\phi(x) = 0$, that is, $f(x) = x$.

11.2 Kakutani's fixed point theorem

In many applications, we need results on the existence of fixed points for correspondences. The most well-known among such applications is probably the existence of Nash equilibrium for finite normal-form games. Kakutani's fixed point theorem is a direct generalization of Brouwer's theorem from functions to correspondences.

Definition 11.2. For a correspondence $\varphi : A \twoheadrightarrow A$, if $x \in \varphi(x)$ for some $x \in A$, then x is a fixed point of φ.

Theorem 11.2 (Kakutani)**.** *Suppose that $A \subset \mathbb{R}^n$ is nonempty, compact and convex, and that $\varphi : A \twoheadrightarrow A$ is an upper hemicontinuous correspondence from A into itself such that $\varphi(x) \subset A$ is nonempty, convex and closed for every $x \in A$. Then $\varphi(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x \in \varphi(x)$.*

Convexity (of both the domain and value) is crucial for both Brouwer and Kakutani's theorems. The simple example below illustrates how Kakutani's theorem could fail when $\varphi(\cdot)$ is not convex valued.

Example 11.2. Consider $\varphi : [0,1] \twoheadrightarrow [0,1]$ defined by

$$\varphi(x) = \begin{cases} \{\frac{3}{4}\}, \text{ if } x \in [0, \frac{1}{2}) \\ \{\frac{3}{4}, \frac{1}{4}\}, \text{ if } x = \frac{1}{2} \\ \{\frac{1}{4}\}, \text{ if } x \in (\frac{1}{2}, 1]. \end{cases}$$

φ has no fixed point; this is because $\varphi(\frac{1}{2})$ is not convex (not an interval).

Remark 11.3. In Kakutani's theorem, we can replace upper hemicontinuity of φ and closed-valuedness with the closed-graph property of φ. Moreover, the theorem also holds when the upper hemicontinuity of φ is replaced by lower hemicontinuity (Michael's theorem).

Remark 11.4. The contraction mapping theorem works for functions, and crucially, it does not require the domain of the function to be convex. For correspondences, Nadler's fixed point theorem generalizes the contraction mapping theorem. It requires the correspondence to be contractive under the Hausdorff metric. Let (X, d) be a complete metric space. A correspondence $\varphi : X \twoheadrightarrow X$ is contractive if there exists $\beta \in (0,1)$ such that $d_H(\varphi(x), \varphi(x')) \leq \beta d(x, x')$ for all $x, x' \in X$.

11.3 Tarski's fixed point theorem

Tarski's fixed point theorem is studied under general order structures, so we need some notations to formally state the theorem.

A partial order is a binary relation $\leq$ on a set X such that $x \leq x$ (reflexive), $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetric), and $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive), for all $x, y, z \in X$. A set X endowed with a partial order $\leq$ is called a **partially ordered set**, or poset, and is denoted by $(X, \leq)$.

Note that unlike total orders, partial orders do not require completeness; that is, for a pair $x, y \in X$, they need not be comparable. The simpliest example of partial order is the " $\leq$ " relation on $\mathbb{R}^k$.

Example 11.3. Set inclusion is a partial order. Fix a set S. For $A, B \in 2^S$, let $A \leq B$ if and only if $A \subset B$.

For any partially ordered set $(X, \leq)$ and $B \subset X, \bar{b}$ is the supremum (or join) of B if $x \leq \bar{b}, \forall x \in B$, and $x \leq b, \forall x \in B$ implies $\bar{b} \leq b$. Similarly, $\underline{b}$ is the infimum (or meet) of B if $\underline{b} \leq x, \forall x \in B$, and $b \leq x, \forall x \in B$ implies $b \leq \underline{b}$. The supremum and infimum of B may not exist, but if either one exists, it is unique. When $B = \{x, y\}$, we often write $\sup B = x \vee y$ and $\inf B = x \wedge y$. We say that $(X, \leq)$ is a **complete lattice** if for all nonempty subset $B \subset X$, both the supremum and infimum of B exist in X. Compact sets in $\mathbb{R}$ under the usual order are simple examples of complete lattices.

Example 11.4. Consider $x, y \in (\mathbb{R}^k, \leq)$. Then $x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_k, y_k\})$ and $x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_k, y_k\})$.

Example 11.5. Consider $(2^S, \subset)$. So $\leq$ is the set inclusion relation $\subset$. For two sets $A, B \in 2^S, A \vee B = A \cup B$ and $A \wedge B = A \cap B$.

The following fixed point theorem relies mainly on order relations and the monotonicity of f, instead of continuity and convexity which are crucial for Brouwer and Kakutani's theorems.

Theorem 11.3 (Tarski)**.** *Suppose $(X, \leq)$ is a complete lattice and $f : X \to X$ is a nondecreasing function ($x \leq y$ implies $f(x) \leq f(y)$). Then the set of fixed points of f is a nonempty complete lattice.*

Proof. Let $\bar{x} = \sup\{x \in X : x \leq f(x)\}$ and $\underline{x} = \inf\{x \in X : f(x) \leq x\}$. Let $E(f)$ be the set of fixed points of f. We first show that $\bar{x} = \sup E(f)$ and $\underline{x} = \inf E(f)$.

Let $B = \{x \in X : x \leq f(x)\}$. Since X is a complete lattice and f is nondecreasing, $\inf X$ exists in X and $\inf X \leq f(\inf X)$. Hence $\inf X \in B$ and B is nonempty. For any $x \in B, x \leq \bar{x}$ and $x \leq f(x)$; hence $x \leq f(x) \leq f(\bar{x})$. So $f(\bar{x})$ is an upper bound of $B, \bar{x} \leq f(\bar{x})$ and $\bar{x} \in B$. Moreover, since $f(\bar{x}) \leq f(f(\bar{x})), f(\bar{x}) \in B$. Since $\bar{x}$ is the supremum of B, together we must have $f(\bar{x}) = \bar{x}$. Lastly, for any fixed point $x' \in E(f), x' \leq x' = f(x')$.

So $x' \in B$. By definition, $x' \leq \bar{x}$. So being a fixed point and the supremum of $B, \bar{x}$ must also be the supremum of $E(f)$. Following similar steps, we can show that $\underline{x} = \inf E(f)$.

It remains to show that $(E(f), \leq)$ is a complete lattice. Let $E' \subset E(f)$ be a subset of fixed points and let $z = \sup E'$. For any $x \in E', x = f(x) \leq z$; hence $x = f(x) \leq f(z)$. Hence $f(z)$ is an upper bound of E', and consequently, $z \leq f(z)$; so we have $z \in B$. The set $\{x \in X : z \leq x\}$ is a complete lattice and f maps it into itself. Due to the arguments above, f must have a minimal fixed point r in this set, which is the least upper bound of E'. If z is also a point in $E(f)$, then $r = z$. Similarly, we can show that the infimum of E' exists in $E(f)$. In sum, $E(f)$ is a complete lattice.

$\square$

This theorem does not hold anymore if we replace nondecreasing function with nonincreasing function. And just as Kakutani generalizes Brouwer's theorem and Nadler generalizes Banach's contraction mapping theorem from functions to correspondences, Zhou (1994) generalizes Tarski's theorem from functions to correspondences.

Example 11.6. Due to Tarski's theorem, any nondecreasing function $f : [0,1]^k \to [0,1]^k$ has a fixed point (note: continuity is relaxed). Moreover, any nondecreasing function $f : [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \to [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ has a fixed point (note: convexity of domain is relaxed).

Example 11.7. Let $f : [0,1] \to [0,1]$ be defined as

$$f(x) = \begin{cases} x, & \text{if } x < 1/2. \\ 1, & \text{if } x \geq 1/2. \end{cases}$$

Then the set of fixed points of f is $E(f) = [0, 1/2) \cup \{1\}$. It is a complete lattice itself (not as as sublattice of $[0,1]$), as any nonempty subset of it has supremum and infimum in it. For example, in $E(f)$, the supremum of the subset $[0, 1/2) \subset E(f)$ is 1.

11.4 Application I: Existence of competitive equilibrium

Consider a pure exchange economy. There are L goods and a set $N = \{1, \ldots, n\}$ of consumers in the economy. Each consumer $i \in N$ has a vector of initial endowments $\omega_i = (\omega_{i1}, \ldots, \omega_{iL}) \in \mathbb{R}^L_+$ and a utility function $u_i : \mathbb{R}^L_+ \to \mathbb{R}$. Each utility function is assumed to be continuous, strictly increasing and strictly quasi-concave.

In a competitive market, each good l has a nonnegative price p_l and the price vector is denoted by $p = (p_1, \dots, p_L)$. Consumers are assumed to be price-takers in the sense that they do not think their consumption can influence prices. The utility maximization problem of consumer i is

$$\max_{x_i \in \mathbb{R}^L_+} u(x_i), \text{ s.t. } p \cdot x_i \leq p \cdot \omega_i.$$

That is, each consumer chooses the best consumption bundle to maximize her utility, given that her income is the market value of her endowment. Given that only relative prices matter for the budget set, it is convenient to normalize the prices and consider only $p \in \Delta^{L-1} \equiv \{p \in \mathbb{R}^L_+ : \sum_{l=1}^{L} p_l = 1\}$.

Definition 11.5. A competitive equilibrium is a tuple $(\bar{x}_1, \dots, \bar{x}_n; \bar{p})$ such that

1. (Individual rationality) For each consumer $i, \bar{x}_i$ solves i's utility maximization problem;
2. (Market clearing) For each good $l, \sum_{i=1}^{n} \bar{x}_{il} = \sum_{i=1}^{n} \omega_{il}$.

To find a competitive equilibrium, the main task is to find a price vector $\bar{p}$ at which the market of all goods clear at the same time. Due to the monotonicity of the utility function, we only need to focus on strict positive prices as whenever any good is free (price is zero), the market for that good will never clear. Then given prices p, we can always solve each consumer i's utility maximization problem and obtain her demand function $x_i(p)$. We know that $p \cdot x_i(p) = p \cdot \omega_i$ (Walras' law). Also, due to Berge's maximum theorem, $x_i(p)$ is continuous in p. (Note: To be mathematically rigorous, we also need to add an exogenous truncation to the budget correspondence to make it compact valued and continuous even at prices where some goods are free.)

Theorem 11.4. *Any pure exchange economy satsifying the assumptions above has at least one competitive equilibrium.*

Proof. Given any market prices p, let the excess demand for good l be $z_l(p) = \sum_{i=1}^{n} x_{il}(p) - \sum_{i=1}^{n} \omega_{il}$. We then only need to show that there exists $\bar{p}$ such that $z_l(\bar{p}) = 0$ for all l. Define a price adjustment process $F : \Delta^{L-1} \to \Delta^{L-1}$ by

$$p_l \to \frac{p_l + \max\{z_l(p), 0\}}{\sum_{k=1}^{L}(p_k + \max\{z_k(p), 0\})} = \frac{p_l + \max\{z_l(p), 0\}}{1 + \sum_{k=1}^{L} \max\{z_k(p), 0\}}.$$

That is, at any normalized price vector p, the price of good l, p_l, will be adjusted by whether good l has excess demand and the magnitude of such excess. The new prices are again normalized.

Since $x_i(p)$ is continuous, F is a continuous function. Due to Brouwer's fixed-point theorem, there exists $\bar{p} \in \Delta^{L-1}$ such that $F(\bar{p}) = \bar{p}$. That is, for each good l,

$$\bar{p}_l = \frac{\bar{p}_l + \max\{z_l(\bar{p}), 0\}}{1 + \sum_{k=1}^{L} \max\{z_k(\bar{p}), 0\}}.$$

Re-organizing it, we have

$$\bar{p}_l \sum_{k=1}^{L} \max\{z_k(\bar{p}), 0\} = \max\{z_l(\bar{p}), 0\}.$$

Multiply both sides by $z_l(\bar{p})$ and then sum up by l,

$$\sum_l \bar{p}_l z_l(\bar{p}) \sum_{k=1}^{L} \max\{z_k(\bar{p}), 0\} = \sum_l z_l(\bar{p}) \max\{z_l(\bar{p}), 0\}.$$

Due to the Walras' law, $\sum_l \bar{p}_l z_l(\bar{p}) = \sum_l \bar{p}_l (\sum_{i=1}^{n} x_{il}(\bar{p}) - \sum_{i=1}^{n} \omega_{il}) = 0$. Plug into the equation above, we have $\sum_l z_l(\bar{p}) \max\{z_l(\bar{p}), 0\} = 0$, which holds only if $\max\{z_l(\bar{p}), 0\} = 0$ for all l, i.e., only if $z_l(\bar{p}) \leq 0$ for all l. Since $\sum_l \bar{p}_l z_l(\bar{p}) = 0$, we must have $z_l(\bar{p}) = 0$ for all l.

□

Example 11.8. Suppose there are two goods and two consumers, with $u_1(x_{11}, x_{12}) = x_{11}^{1/4} x_{12}^{3/4}$ and $u_2(x_{21}, x_{22}) = x_{21}^{3/4} x_{22}^{1/4}$. Consumers' endowments are $\omega_1 = (2, 1)$ and $\omega_2 = (2, 2)$. For convenience, let $p = (1, p_2)$. Then consumer 1's utility maximization problem is

$$\max x_{11}^{1/4} x_{12}^{3/4}, \text{ s.t. } x_{11} + p_2 x_{12} \leq 2 \times 1 + 1p_2.$$

Solving this problem gives us

$$x_{11}(p) = \frac{2+p_2}{4}, x_{12}(p) = \frac{3(2+p_2)}{4p_2}.$$

Likewise, for consumer 2, we have

$$x_{21}(p) = \frac{3(2+2p_2)}{4}, x_{22}(p) = \frac{2+2p_2}{4p_2}.$$

The market clearing condition of good 1,

$$x_{11}(p) + x_{21}(p) = \omega_{11} + \omega_{21} = 2 + 2,$$

which implies $p_2 = \frac{8}{7}$.

Due to Walras' law, when the market for good 1 clears, so will that of good 2. The competitive equilibrium is then

$$(\bar{x}_1, \bar{x}_2; \bar{p}) = ((\frac{11}{14}, \frac{33}{16}), (\frac{45}{14}, \frac{15}{16}); (1, \frac{8}{7})).$$

Next, we show the welfare properties of competitive market. Consumers in the market exchange their endowments to improve their utilities, and the proportion of exchange is coordinated by prices. The first welfare theorem states that the allocation of competitive equilibrium is Pareto efficient, as it exhausts all exchange opportunities among consumers.

An allocation $(x_1, \ldots, x_n)$ is said to be feasible if $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} \omega_i$. A feasible allocation is Pareto efficient for consumers if it is not Pareto dominated by any other feasible allocation.

Theorem 11.5. *Any competitive equilibrium allocation $(\bar{x}_1, \ldots, \bar{x}_n)$ is Pareto efficient.*

Proof. Let the equilibrium price vector be $\bar{p}$. Suppose instead the equilibrium allocation $(\bar{x}_1, \ldots, \bar{x}_n)$ is not Pareto efficient. Then it is Pareto dominated by some feasible allocation $(y_1, \ldots, y_n)$. That is, $u_i(y_i) \geq u_i(\bar{x}_i)$ for all consumer i and $u_j(y_j) > u_j(\bar{x}_j)$ for at least one consumer j. Since for each $i, \bar{x}_i$ is optimal in i's budget set, we know that $\bar{p} \cdot y_i \geq \bar{p} \cdot \omega_i$ for all i and $\bar{p} \cdot y_j > \bar{p} \cdot \omega_j$ for at least one j. As a result, $\bar{p} \cdot \sum_{i=1}^{n} y_i > \bar{p} \cdot \sum_{i=1}^{n} \omega_i$, which contradicts with y's feasibility.

□

As we can imagine, market exchange does not ensure fairness. The second welfare theorem argues that unfairness of the market exchange outcome comes from the unfairness of endowments, instead of the market mechanism. If any Pareto efficient allocation is desirable, the planner can always achieve it through market exchange by properly adjusting endowments or transfering money across consumers.

Theorem 11.6. *Suppose* $(\bar{x}_1, \ldots, \bar{x}_n)$ *is Pareto efficient. Then there exists price vector* p *and transfer vector* $(t_1, \ldots, t_n)$ *with* $\sum_i t_i = 0$ *such that for all* $i, \bar{x}_i$ *solves*

$$\max_{x_i \in \mathbb{R}^L_+} u_i(x_i), \ \ s.t. \ p \cdot x_i \leq p \cdot \omega_i + t_i.$$

Proof. Suppose $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ is Pareto efficient. Then $\sum_i \bar{x}_i = \sum_i \omega_i$. Let

$$U(\bar{x}) = \{y = (y_1, \ldots, y_n) : y_i \succ_i \bar{x}_i, \forall i\}.$$

Then the set $\{\sum_i y_i : y \in U(\bar{x})\}$ is convex and does not contain $\sum_i \bar{x}_i$. Due to the hyperplane separation theorem, there exists $p \in \mathbb{R}^L_+$ (strictly positive as u is increasing) such that for all $y \in U(\bar{x})$,

$$p \cdot \sum_i y_i > p \cdot \sum_i \bar{x}_i. \tag{11.1}$$

Note that if $y_1 \succ \bar{x}_1$, then $(y_1, x_2 + \varepsilon, \ldots, x_n + \varepsilon) \in U(\bar{x})$. Plug this into Equation 11.1 and let $\varepsilon \to 0$, we have $p \cdot y_1 \geq p \cdot \bar{x}_1$. More generally, $p \cdot y_i \geq p \cdot \bar{x}_i$ for all i. That is, at price vector p, any bundle y_i better than $\bar{x}_i$ costs at least as much as $\bar{x}_i$. By the continuity of u_i, for small enough $\varepsilon, y_i - \varepsilon$ must also cost at least as much, hence y_i must cost strictly more.

Lastly, for each i, let $t_i = p \cdot \bar{x}_i - p \cdot \omega_i$ so that with transfer, $\bar{x}_i$ becomes affordable at p. Together, for each consumer $i, \bar{x}_i$ is affordable under transfer but any bundle better than it is not, hence it solves i's utility maximization problem under transfer.

□

11.5 Application II: Existence of Nash equilibrium

In games, the payoff to each player depends not only on his own action, but also on the actions taken by his opponents. Take two-player games for example, in order to make the right decision, a player needs to conjecture about the other player's action, which in turn depends on the other player's conjecture about her own action, which further depends on the other player's conjecture about her conjecture on the other's action, and so on. Such interactive thinking is the major feature of game theory.

Definition 11.6 (Normal-form game)**.** A finite strategic-form (normal-form) non-cooperative game is a tuple $G = \{u_i, A_i\}_{i \in N}$, where

1. $N = \{1, \ldots, n\}$ denotes the finite set of players.
2. A_i is the finite set of actions (strategies) of player i, with a generic element a_i.
3. $u_i : A_1 \times \cdots \times A_n \to \mathbb{R}$ is the payoff function of player i.

Let $A \equiv \times_{i \in N} A_i$ and $A_{-i} = \times_{j \neq i} A_j$. Each $a = (a_1, \ldots, a_n) \in A$ is a profile of players' actions, and each $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A_{-i}$ is a profile of i's opponents' actions. As a convention, we write $a = (a_i, a_{-i})$. Each action $a_i \in A_i$ is a pure strategy of player i. A mixed strategy involves randomization over actions, and is defined as a probability distribution over pure strategies. Formally, let

$$\Delta A_i = \{\sigma_i \in [0,1]^{A_i} : \sum_{a_i \in A_i} \sigma_i(a_i) = 1\}$$

denote the set of probability distributions on A_i. Then each $\sigma_i \in \Delta A_i$ is a mixed strategy of player i. Given any mixed strategy profile $\sigma = (\sigma_i, \sigma_{-i}) \in \times_{i \in N} \Delta A_i$, the payoff to player i is

$$u_i(\sigma_i, \sigma_{-i}) = \sum_{a \in A} \sigma(a) u_i(a_i, a_{-i}),$$

where $\sigma(a) = \prod_{j \in N} \sigma_j(a_j)$ is the probability that action profile a is played under the mixed strategy profile σ.

Definition 11.7 (Nash equilibrium)**.** Fix a strategic-form game $G = \{u_i, A_i\}_{i \in N}$.

1. A pure action profile $a = (a_1, \ldots, a_n)$ is a **pure strategy Nash equilibrium** of G if

$$u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \forall a'_i \in A_i, i \in N.$$

2. A mixed strategy profile $\sigma = (\sigma_1, \ldots \sigma_n)$ is a **mixed strategy Nash equilibrium** of G if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}), \forall \sigma'_i \in \Delta A_i, i \in N.$$

Inuitively, an action profile (or a mixed strategy profile) forms a Nash equilibrium if, given that all other players play the actions specified by this profile, nobody has the incentive to deviate from the specified action.

Example 11.9 (Battle of the Sexes)**.** Consider the *Battle of the Sexes* game played by a man and a woman, both of whom choose between watching *Ballet* (B) and *Fight* (F). The left table gives the payoff matrix of the game (with woman choosing the row), and the right table gives the probability that each action profile in $\{B, F\}^2$ is played under players' mixed strategies $\sigma_1 = \alpha B + (1-\alpha)F$ and $\sigma_2 = \beta B + (1-\beta)F$.

	B	F
B	$2, 1$	$0, 0$
F	$0, 0$	$1, 2$

	βB	$(1-\beta)F$
αB	$\alpha\beta$	$\alpha(1-\beta)$
$(1-\alpha)F$	$(1-\alpha)\beta$	$(1-\alpha)(1-\beta)$

There are two pure strategy Nash equilibria: (B, B) and (F, F). There is one mixed strategy equilibrium: $(\frac{2}{3}B + \frac{1}{3}F, \frac{1}{3}B + \frac{2}{3}F)$.

Example 11.10 (Matching pennies)**.** Two players simutaneously announce *Head* (H) or *Tail* (T). Player 1 wins if their announcements match, and player 2 wins if their announcements mismatch.

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

This game has no pure strategy equilibrium, but has a mixed strategy equilibrium $(\frac{1}{2}H + \frac{1}{2}T, \frac{1}{2}H + \frac{1}{2}T)$.

Theorem 11.7 (Nash, 1950)**.** *Every finite strategic-form game has a mixed-strategy equilibrium.*

Proof. For any mixed strategy profile $\sigma = (\sigma_1, \dots, \sigma_n)$, for each $i \in N$, define player i's best response correspondence $BR_i : \times_{j\in N}\Delta A_j \twoheadrightarrow \Delta A_i$ as

$$BR_i(\sigma) = \arg\max_{\sigma_i' \in \Delta A_i} u_i(\sigma_i', \sigma_{-i}).$$

Since u_i is continuous (actually linear) in σ and ΔA_i is convex and compact, by Berge's maximum theorem, $BR_i(\sigma)$ is upper hemicontinuous; it is also nonempty, closed and convex valued. Let $BR : \times_{i\in N}\Delta A_i \twoheadrightarrow \times_{i\in N}\Delta A_i$ be defined as $BR = (BR_1, \dots, BR_n)$. Given each $\sigma, BR(\sigma)$ generates the best response mixed strategy profile. So

$$BR(\sigma) = (BR_1(\sigma), \dots, BR_n(\sigma)).$$

Due to the product structure of $BR(\cdot)$, properties of $BR_i(\cdot)$ carry over to $BR(\cdot)$. That is, $BR(\cdot)$ is also nonempty, closed and convex valued. Therefore, by Kakutani's fixed point theorem, $BR(\cdot)$ has a fixed point. That is, there exists $\sigma \in \times_{i\in N}\Delta A_i$ such that $\sigma \in BR(\sigma)$. According to the definition of $BR(\cdot), \sigma_i \in BR_i(\sigma)$ for all i, hence σ is a mixed strategy Nash equilibrium.

□

Example 11.11 (Non-existence of NE in infinite games)**.** Let $N = \{i, j\}$ and $A_i = A_j = \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. The payoff functions are

$$u_i = \begin{cases} 1 \text{ if } a_i > a_j \\ 0 \text{ if } a_i \leq a_j. \end{cases}$$

Intuitively, the players announce numbers simultaneously, and a player wins a dollar if and only if she announces a larger number. No pure or mixed NE.

11.6 Application III: Stable matchings as fixed points

A marriage market is a tuple $(M, W, (\succ_m)_{m\in M}, (\succ_w)_{w\in W})$, where M is a finite set of men and W is a finite set of women; for each man $m \in M, \succ_m$ is his strict preference over $W \cup \{\emptyset\}$, and for each woman $w \in W, \succ_w$ is her strict preference over $M \cup \{\emptyset\}$, where $\emptyset$ stands for remaining single. If a man m prefers woman w to woman w', we write $w \succ_m w'$, and if he prefers remaining single to marrying woman w, we write $\emptyset \succ_m w$. Let $\succeq$ be the weak preference associated with $\succ$.

A **pre-matching** ν maps $M \cup W$ into $M \cup W \cup \{\emptyset\}$, such that $\nu(m) \in W \cup \{\emptyset\}$ and $\nu(w) \in M \cup \{\emptyset\}$, and a pre-matching μ is a **matching** if $\mu(m) = w \Leftrightarrow \mu(w) = m$. Let V be the set of pre-matchings. The difference between pre-matching and matching is that a pre-matching is not necessarily feasible: it pre-matches each man/woman with the one that he/she proposes to, in spite of the fact that the other party may not wish to accept him/her.

Definition 11.8 (Stability)**.** A matching μ is **stable** if

1. (individually rational) for all m and $w, \mu(m) \succeq_m \emptyset$ and $\mu(w) \succeq_w \emptyset$;

2. (no blocking pair) there is no (m, w) such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$.

We say that a stable matching μ is man-optimal if it weakly Pareto dominates all other stable matchings from men's perspective. Formally, μ is the man-optimal stable matching if for any stable matching $\mu', \mu(m) \succeq_m \mu'(m)$ for all $m \in M$. The famous man-proposing **Gale-Shapley Deferred Acceptance Algorithm (DA)** proves the existence of the man-optimal stable matching by directly producing it through a propose-reject process. The algorithm operates as follows:

Step 1 Each man proposes to his most favorite woman, and each woman (if facing proposals) tentatively accepts the best proposer according to her preference and rejects all other proposers.

Step $k, k \geq 2$ Each man, if rejected in the previous step, proposes to his next favorite woman, and each woman (if facing proposals) tentatively accepts the best proposer according to her preference, among both new proposers and the previously accepted man, and reject all others.

The algorithm stops at the step when no man is rejected.

This matching produced by this algorithm is the best stable matching for men but the worst stable matching from women's perspective. Moreover, this algorithm is strategyproof, in the sense that no man has incentive to misreport his preference (but women may misreport). (Note: stability is straightforward, but proofs of the other results are nontrivial. If interested, please refer to Roth and Sotomayor, *Two-sided Matching*, 1990.) Among many of the applications of DA, the most influential one is school choice, where students propose to schools, and the algorithm produces the student-optimal stable matching.

Similarly, the woman-proposing DA algorithm produces the woman-optimal stable matching. In general, a marriage market has many stable matchings, other than the two extremal ones produced by the algorithm. To study the structure of the whole set of stable matchings, we use tools related to Tarski's fixed point theorem.

Define $T : V \to V$ such that for all pre-matching $\nu \in V, m \in M$ and $w \in W$,

$$\begin{aligned} T(\nu)(m) &= \sup_{\succ_m}\{w \in W : m \succeq_w \nu(w)\} \cup \{m\}, \\ T(\nu)(w) &= \sup_{\succ_w}\{m \in M : w \succeq_m \nu(m)\} \cup \{w\}, \end{aligned}$$

where the supremum is taken according to the respective agent's preference. Since we assume strict preferences, $T(\nu)$ is always single valued and is thus a pre-matching; hence T is a function from V to V.

Given any pre-matching $v, T(\nu)(m)$ matches each man $m \in M$ with his most favorite woman among women who prefer him to their respective pre-match under ν, and $T(\nu)(w)$ matches each woman $w \in W$ with her most favorite man among men who prefer her to their respective pre-match under ν. So each time we apply T on a pre-matching ν, in a sense, the new pre-matching $T\nu$ becomes more feasible.

Lemma 11.8. *A pre-matching ν is a stable matching if and only if it is a fixed point of T.*

The proof of this lemma is left to you as an exercise. We now define a partial order for the set of pre-matchings. Let $\nu \leq \nu'$ if $\nu'(m) \succeq \nu(m)$ and $\nu(w) \succeq \nu'(w)$, for all m, w. (Note the reversal in the definition.) So $\nu \vee \nu'$ ($\nu \wedge \nu'$, respectively) is the pre-matching that matches each man m with the better (worse) one in $\{\nu(m), \nu'(m)\}$ and matches each woman w with the worse (better) one in $\{\nu(w), \nu'(w)\}$. Then $(V, \leq)$ is a complete lattice. Moreover, if $\nu \leq \nu'$, then $T\nu \leq T\nu'$; hence T is nondecreasing.

As a direct result of Tarski's fixed point theorem, we obtain the lattice structure of the set of stable matchings.

Theorem 11.9. *The set of fixed points of T, that is, the set of stable matchings for the marriage market, is a nonempty complete lattice.*

The lattice structure of the set of stable matchings implies that the join and meet of two stable matchings are also stable, and there exists an optimal stable matching for men (the supremum of the lattice) and an optimal stable matching for women (the infimum of the lattice). The man-optimal matching can be obtained by iteratively applying T, starting with the pre-matching ν such that $\nu(m) = \sup_{\succ_m}\{W \cup \{\emptyset\}\}$ and $\nu(w) = \{w\}$. You should be able to see that this process is exactly the DA algorithm. The woman-optimal matching can be obtained in the reverse way.