

Robustifying the Permanent Income Model with Rational Inattention^{*}

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December 24, 2011

Abstract

In this paper we examine implications of model uncertainty due to robustness (RB) for consumption decisions, aggregate savings, and welfare under limited information-processing capacity (rational inattention or RI) in an otherwise standard permanent income model with filtering. We first show that RB and risk-sensitivity (RS) are observationally equivalent (OE) under imperfect information due to RI in the sense that they lead to the same consumption and saving decisions if consumers use the *regular* Kalman filter to extract signals. Second, we find that once allowing RS consumers to use the risk-sensitive filter and RB consumers to use the robust Kalman filter to extract signals, the absolute and linear OE between RB and RS no longer holds; instead, we find a conditional and nonlinear OE between RB and RS. Furthermore, we find that in the filtering problem, either a stronger preference for robustness in the Kalman gain, a stronger risk-sensitive preference, or higher channel capacity can increase the Kalman gain. Finally, we explore how RB in the filtering problem interacts with RI and affects consumption dynamics, aggregate savings, and the welfare costs of uncertainty.

JEL Classification Numbers: *C61, D81, E21.*

^{*}We would like to thank Richard Dennis, Larry Epstein, Ken Kasa, Tasos Karantounias, Jun Nie, Kevin Salyer, Tom Sargent, Martin Schneider, Chris Sims, Laura Veldkamp, and Mirko Wiederholt, as well as seminar and conference participants at UC Davis, The Hong Kong University of Science and Technology, University of Toyko, the conference on “Putting Information Into (or Taking it out of) Macroeconomics” organized by LAEF of UCSB, and the 2011 Summer Meeting of Econometric Society for helpful discussions and comments. Luo thanks the General Research Fund (GRF) in Hong Kong and the HKU seed funding program for basic research for financial support. All errors are the responsibility of the authors.

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Keywords: Robust Kalman Gain, Risk-sensitive Filtering, Rational Inattention, Observational Equivalence, Welfare Costs.

1 Introduction

Hansen and Sargent (1995, 2007) first introduce robustness (RB, a concern for model misspecification) into linear-quadratic (LQ) economic models. In robust control problems, agents do not know the true model generating the data and are concerned about the possibility that their model (denoted the approximating model) is misspecified; consequently, they choose optimal decisions as if the subjective distribution over shocks was chosen by an evil nature in order to minimize their expected utility.¹ Robustness models produce precautionary savings but remain within the class of LQ models, which leads to analytical simplicity.² A second class of models that produces precautionary savings but remains within the class of LQ models is the risk-sensitive (RS) model of Hansen and Sargent (1995) and Hansen, Sargent, and Tallarini (1999) (henceforth, HST). In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by overweighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states. As shown in Hansen and Sargent (2007), the risk-sensitivity preference can be used to characterize robustness as they lead to the same consumption, saving, and welfare.³

In Luo and Young (2010a), we study the properties of risk-sensitive and robust LQ permanent income models with rational inattention (RI), and show that RS, RB, and the discount factor are observationally equivalent in the sense that RB-RI, RS-RI, and RI models possess combinations of parameters such that their implied consumption-savings rules are the same. Sims (2003) first introduce rational inattention into economics and argued that it is a plausible method for introducing sluggishness, randomness, and delay into economic models. In his formulation agents have finite Shannon channel capacity, limiting their ability to process signals about the true state of the world. As a result, an impulse to the economy induces only gradual responses by individuals, as their limited capacity requires many periods to discover just how much the state has moved. The key assumption in Luo and Young (2010) is that agents with finite capacity

¹The solution to a robust decision-maker's problem is the equilibrium of a max-min game between the decision-maker and nature.

²See Hansen and Sargent (2010) for a recent survey on robustness.

³It is worth noting that although both RB (or RS) and CARA preferences (i.e., Caballero 1990 and Wang 2003) increase the precautionary savings premium via the intercept terms in the consumption functions, they have distinct implications for the marginal propensity to consume out of permanent income (MPC). Specifically, CARA preferences do not alter the MPC relative to the LQ case, whereas RB or RS increases the MPC. That is, under RB, in response to a negative wealth shock, the consumer would choose to reduce consumption more than that predicted in the CARA model (i.e., save more to protect themselves against the negative shock).

distrust their budget constraint, but still use an *ordinary* Kalman filter to estimate the true state; in this case, a distortion to the mean of permanent income is introduced to represent possible model misspecification. However, this case ignores the effect of the RI-induced noise on the demand for robustness. In other words, since RI introduces additional uncertainty, the endogenous noise due to finite capacity, into economic models, RI by itself creates an additional demand for robustness.

In this paper we first construct an RB-RI permanent income model with two types of concerns about model misspecification: (i) concerns about the disturbances to the perceived permanent income (the disturbances here include both the fundamental shock and the RI-induced noise shock) and (ii) concerns about the Kalman gain. For ease of presentation, we will refer to the first type of robustness as Type I and the second as Type II. Furthermore, in the RS permanent income model with information imperfections due to RI, the classical Kalman filter that extremizes the expected value of a certain quadratic objective function is no longer optimal in this risk-sensitive LQ setting. The reason is that in the risk-sensitive LQ problem the objective function is an exponential-quadratic function; consequently, the risk-sensitive filter is more suitable than the classical Kalman filter (the conditional mean estimator) for estimating the imperfectly observed state.⁴ After solving the RB and RS models with suitable filters, we establish the observational equivalence (OE) conditions between RB and RS. We find that the absolute and linear OE between RB and RS established in Hansen and Sargent (2007) and Luo and Young (2010) no longer holds, we instead have conditional and highly nonlinear OE between RB and RS.

We then examine how different types of robustness affects optimal consumption, precautionary savings, and welfare costs of uncertainty via interacting with finite capacity.⁵ Specifically, we show that given finite capacity, the two types of robustness have *opposing* impacts on the marginal propensity to consume out of perceived permanent income (MPC) and precautionary savings. For Type I robustness, since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more. As for Type II robustness, an increase in the strength of this effect increases the Kalman gain, which leads to lower total uncertainty about the true level of permanent income and then low precautionary savings. In addition, the strength of the precautionary effect is

⁴See Speyer, Fan, and Banavar (1992) and Banavar and Speyer (1998).

⁵Note that since RB and RS are OE, we can obtain the similar results about the effects of RS on consumption, saving, and welfare.

positively related to the amount of this uncertainty that always increases as finite capacity gets smaller. Using the explicit expression for consumption dynamics, we also show that increasing Type II robustness increases the robust Kalman filter gain and thus leads to less relative volatility of consumption to income (less smooth consumption process). In contrast, Type I robustness increases the relative volatility of consumption by increasing the MPC out of changes in permanent income. This mechanism is similar to that examined in Luo and Young (2010a). In an economy with a continuum of agents whose survival probability is p , we find that in the steady state equilibrium, RB and RI not only affects the level of aggregate precautionary savings, but also affects the level of non-precautionary wealth. Specifically, RB and RI reduce the level of non-precautionary wealth and increases the level of precautionary savings; as a result, they increase the relative importance of precautionary savings in aggregate wealth. Finally, using the established OE conditions, we show that under the OE, robustness (or risk-sensitivity) and the discount factor lead to different welfare costs of uncertainty. Specifically, we show that given finite capacity, the welfare losses for RB-RI agents are decreasing with both Type I robustness and Type II robustness.

The remainder of the paper is organized as follows. Section 2 presents the robustness version of the RI permanent income model with the regular Kalman filter, and discusses the observational equivalence result between robustness, risk-sensitivity and the discount factor. Section 3 introduces risk-sensitive filtering and establish the OE between RS and RB in this case. Section 4 introduces robust Kalman gain and studies how two types of robustness, concerns about the fundamental shock and the RI-induced noise and the robust Kalman gain, affect consumption and precautionary savings. We then establish the OE between RB and RS in this case. Section 5 examines how robust filtering and risk-sensitive filtering affect consumption dynamics, aggregate savings and welfare. Section 6 concludes.

2 Robust Control and Filtering under Rational Inattention

2.1 A Rational Inattention Version of the Standard Permanent Income Model

In this section we consider a rational inattention (RI) version of the standard permanent income model. In the standard permanent income model (Hall 1978, Flavin 1981), households solve the dynamic consumption-savings problem

$$v(s_0) = \max_{\{c_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \quad (1)$$

subject to

$$s_{t+1} = Rs_t - c_t + \zeta_{t+1}, \quad (2)$$

where $u(c_t) = -\frac{1}{2}(c_t - \bar{c})^2$ is the period utility function, $\bar{c} > 0$ is the bliss point, c_t is consumption,

$$s_t = w_t + \frac{1}{R} \sum_{j=0}^{\infty} R^{-j} E_t[y_{t+j}] \quad (3)$$

is permanent income, i.e., the expected present value of lifetime resources, consisting of financial wealth plus human wealth,

$$\zeta_{t+1} = \frac{1}{R} \sum_{j=t+1}^{\infty} \left(\frac{1}{R}\right)^{j-(t+1)} (E_{t+1} - E_t)[y_j]; \quad (4)$$

is the time $(t+1)$ innovation to permanent income, w_t is cash-on-hand (or market resources), y_t is a general income process with Gaussian white noise innovations, β is the discount factor, and $R > 1$ is the constant gross interest rate at which the consumer can borrow and lend freely.⁶ Note that when y follows the following AR(1) process with the persistence coefficient $\rho \in [0, 1]$, $y_{t+1} - \bar{y} = \rho(y_t - \bar{y}) + \varepsilon_{t+1}$, $\zeta_{t+1} = \varepsilon_{t+1}/(R - \rho)$, where ε_{t+1} is iid with mean 0 and variance ω^2 . For the rest of the paper we will restrict attention to points where $c_t < \bar{c}$, so that utility is increasing and concave. This specification follows that in Hall (1978) and Flavin (1981) and implies that optimal consumption is determined by permanent income:

$$c_t = \left(R - \frac{1}{\beta R}\right) s_t - \frac{1}{R-1} \left(1 - \frac{1}{\beta R}\right) \bar{c}. \quad (5)$$

We assume for the remainder of this section that $\beta R = 1$, since this setting is the only one that implies zero drift in consumption. Under this assumption the model leads to the well-known random walk result of Hall (1978):

$$\Delta c_{t+1} = (R - 1) \zeta_{t+1}; \quad (6)$$

the change in consumption depends neither on the past history of labor income nor on anticipated changes in labor income. We also point out the well-known result that the standard PIH model with quadratic utility implies the certainty equivalence property holds; thus, uncertainty has no impact on optimal consumption, so that there is no precautionary saving.

Following Sims (2003) and Luo (2008), incorporating RI into the above otherwise standard permanent income model leads to the following consumption rule:

$$c_t = \left(R - \frac{1}{\beta R}\right) \hat{s}_t - \frac{1}{R-1} \left(1 - \frac{1}{\beta R}\right) \bar{c}, \quad (7)$$

⁶We only require that y_t and R are such that permanent income is finite.

where $\widehat{s}_t = E_t[s_t]$ is the perceived state (the conditional mean of the state). \widehat{s}_t is governed by the following Kalman filtering equation

$$\widehat{s}_{t+1} = (1 - \theta)(R\widehat{s}_t - c_t) + \theta(s_{t+1} + \xi_{t+1}), \quad (8)$$

where $\theta = \Sigma/\Lambda = 1 - 1/\exp(2\kappa) \in [0, 1]$ is the constant optimal weight on any new observation,⁷ κ is the consumer's channel capacity, $\Sigma = \sigma^2 = \text{var}[s_t|\mathcal{I}_t] = \frac{\omega_\zeta^2}{\exp(2\kappa) - R^2}$ is the conditional variance of s_t , ξ_{t+1} is the iid endogenous noise with $\Lambda = \lambda^2 = \text{var}[\xi_{t+1}] = \frac{[\omega_\zeta^2 + R^2\Sigma]\Sigma}{\omega_\zeta^2 + (R^2 - 1)\Sigma}$, and given $s_0 \sim N(\widehat{s}_0, \Sigma)$. Note that after substituting (2) into (8), we have an alternative expression of the regular Kalman filter:

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \eta_{t+1}, \quad (9)$$

where

$$\eta_{t+1} = \theta R(s_t - \widehat{s}_t) + \theta(\zeta_{t+1} + \xi_{t+1}) \quad (10)$$

is the innovation to the mean of the distribution of perceived permanent income,

$$s_t - \widehat{s}_t = \frac{(1 - \theta)\zeta_t}{1 - (1 - \theta)R \cdot L} - \frac{\theta\xi_t}{1 - (1 - \theta)R \cdot L}. \quad (11)$$

and $E_t[\eta_{t+1}] = 0$ because the expectation is conditional on the perceived signals and inattentive agents cannot perceive the lagged shocks perfectly.⁸

In the next section, we will discuss alternative ways to robustify this RI-PIH model and their different implications for consumption, precautionary savings, and the welfare costs of uncertainty. The RB-RI model proposed here encompasses the hidden state (HS) model discussed in Hansen, Sargent, and Wang (2002) and Hansen and Sargent (2007b); the main difference is that agents in the RB-RI model cannot observe the entire state vector perfectly, whereas agents in the RB-HS model can observe some part of the state vector (in particular, the part they control).

2.2 Concerns about the Fundamental Shock and the Noise Shock

As shown in Hansen and Sargent (2007), we can robustify the permanent income model by assuming agents with finite capacity distrust their model generating the data (i.e., their income process), but still use an ordinary Kalman filter to estimate the true state. In this model, a distortion to the mean of income is introduced to represent possible model misspecification.

⁷That is, θ measures how much new information is transmitted each period or how much uncertainty is removed upon the receipt of a new signal.

⁸For the filter to converge to the steady state we need $\kappa > \log(R) \approx R - 1$.

Without the concern for model misspecification, the consumer has no doubts about the probability model used to form the conditional expectation of current and future income.

Following Luo and Young (2010), the robust permanent income (PI) problem with inattentive consumers is formulated as follows:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t} \min_{\nu_t} \left\{ -\frac{1}{2} (\bar{c} - c_t)^2 + \beta E_t [\vartheta \nu_t^2 + \widehat{v}(\widehat{s}_{t+1})] \right\} \quad (12)$$

subject to the budget constraint

$$s_{t+1} = R s_t - c_t + \omega_\zeta \nu_t + \zeta_{t+1}, \quad (13)$$

and the regular Kalman filter equation

$$\widehat{s}_{t+1} = (1 - \theta) (R \widehat{s}_t - c_t + \omega_\zeta \nu_t) + \theta (s_{t+1} + \xi_{t+1}), \quad (14)$$

where s_t is permanent income, c_t is consumption, ν_t is the worst-possible case shock, ϑ is the parameter measuring the preference for RB, θ is the Kalman filter gain, and ξ_{t+1} is the iid endogenous noise due to RI. After substituting (13) into (14), the Kalman filter equation can be rewritten as

$$\widehat{s}_{t+1} = R \widehat{s}_t - c_t + \omega_\zeta \nu_t + \eta_{t+1}. \quad (15)$$

where η_{t+1} is given in . The following proposition summarizes the solution to the above RB-RI model when $\beta R = 1$.

Proposition 1 *Given ϑ and θ , the consumption function is*

$$c_t = \frac{R - 1}{1 - \Pi} \widehat{s}_t - \frac{\Pi}{1 - \Pi} \bar{c}, \quad (16)$$

where

$$\Pi = \frac{R \omega_\zeta^2}{2 \vartheta} \in (0, 1) \in (0, 1). \quad (17)$$

Proof. See Appendix 7.1. ■

However, it is clear that the Kalman filter under RI, (9), is not only affected by the fundamental shock (ζ_{t+1}), but also affected by the endogenous noise (ξ_{t+1}) induced by finite capacity; these noise shocks could be another source of the demand for robustness. We therefore need to consider this demand for robustness in the RB-RI model. By adding the additional concern for robustness developed here, we are able to strengthen the effects of robustness on decisions.⁹

⁹Luo, Nie, and Young (2010) use this approach to study the joint dynamics of consumption, income, and the current account.

Specifically, we assume that the agent thinks that (9) is the approximating model. Following Hansen and Sargent (2007), we surround (9) with a set of alternative models to represent a preference for robustness:

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\eta \nu_t + \eta_{t+1}. \quad (18)$$

Under RI the innovation η_{t+1} , (10), that the agent distrusts is composed of two MA(∞) processes and includes the entire history of the exogenous income shock and the endogenous noise, $\{\zeta_{t+1}, \zeta_t, \dots, \zeta_0; \xi_{t+1}, \xi_t, \dots, \xi_0\}$. The difference between (15) and (18) is the third term; in (15) the coefficient on ν_t is ω_ζ while in (18) the coefficient is ω_η ; note that with $\theta < 1$ and $R > 1$ it holds that $\omega_\zeta < \omega_\eta$.

The optimizing problem for this RB-RI model can be formulated as follows:

$$\widehat{v}(\widehat{s}_t) = \max_{c_t} \min_{\nu_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta E_t [\vartheta \nu_t^2 + \widehat{v}(\widehat{s}_{t+1})] \right\} \quad (19)$$

subject to (18). (19) is a standard dynamic programming problem and can be easily solved using the standard procedure. The following proposition summarizes the solution to the RB-RI model.

Proposition 2 *Given ϑ and θ , the consumption function under RB and RI is*

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (20)$$

the mean of the worst-case shock is

$$\omega_\eta \nu_t = \frac{(R-1)\Pi}{1-\Pi} \widehat{s}_t - \frac{\Pi}{1-\Pi} \bar{c},$$

and \widehat{s}_t is governed by

$$\widehat{s}_{t+1} = \rho_s \widehat{s}_t + \eta_{t+1}. \quad (21)$$

where $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0, 1)$,

$$\Pi = \frac{R\omega_\eta^2}{2\vartheta} > 0, \quad (22)$$

$$\omega_\eta^2 = \text{var}[\eta_{t+1}] = \frac{\theta}{1 - (1-\theta)R^2} \omega_\zeta^2,$$

where η_{t+1} is defined in (10), and $\theta(\kappa) = 1 - 1/\exp(2\kappa)$.

Proof. See Appendix 7.3. ■

Equations (20) and (22) determine the effects of model uncertainty due to RB and state uncertainty due to RI on the marginal propensity to consume and the precautionary saving

premium. We now can compare these effects from proportionate shifts in ϑ governing RB and θ governing RI. Specifically, the marginal effects on Π from a proportionate increase in ϑ and θ are given by

$$\frac{\partial \Pi}{\partial \vartheta} \vartheta = -\frac{R\omega_\eta^2}{2\vartheta} \quad \text{and} \quad \frac{\partial \Pi}{\partial \theta} \theta = \frac{R}{2\vartheta} \frac{\theta(1-R^2)}{[1-(1-\theta)R^2]^2} \omega_\zeta^2,$$

respectively. Therefore, the marginal rate of transformation between proportionate changes in ϑ and θ is given by

$$\text{mrt} = -\frac{(\partial \Pi / \partial \vartheta) \vartheta}{(\partial \Pi / \partial \theta) \theta} = \frac{1 - (1 - \theta) R^2}{1 - R^2}. \quad (23)$$

This result gives the proportionate increase in θ that compensates, at the margin, for an proportionate decrease in ϑ —in the sense of preserving the same impact on the consumption function. Equation (23) shows that this compensating change depends only on the interest rate and the degree of inattention measured by θ .

2.3 RS-RI Model with the Regular Kalman Filter and Observational Equivalence Between RS and RB

Risk-sensitivity (RS) was first introduced into the LQ-Gaussian framework by Jacobson (1973) and extended by Whittle (1981, 1990). Exploiting the recursive utility framework of Epstein and Zin (1989), Hansen and Sargent (1995) introduce discounting into the RS specification and show that the resulting decision rules are time-invariant. In the RS model agents effectively compute expectations through a distorted lens, increasing their effective risk aversion by over-weighting negative outcomes. The resulting decision rules depend explicitly on the variance of the shocks, producing precautionary savings, but the value functions are still quadratic functions of the states.¹⁰ In HST (1999) and Hansen and Sargent (2007), they interpret the RS preference in terms of a concern about model uncertainty (robustness or RB) and argue that RS introduces precautionary savings because RS consumers want to protect themselves against model specification errors.

Following Luo and Young (2010), we formulate an RI version of risk-sensitive control based on recursive preferences with an exponential certainty equivalence function as follows:

$$\hat{v}(\hat{s}_t) = \max_{c_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta \mathcal{R}_t [\hat{v}(\hat{s}_{t+1})] \right\} \quad (24)$$

subject to the Kalman filter equation 9. The distorted expectation operator is now given by

$$\mathcal{R}_t [\hat{v}(\hat{s}_{t+1})] = -\frac{1}{\alpha} \log E_t [\exp(-\alpha \hat{v}(\hat{s}_{t+1}))],$$

¹⁰Formally, one can view risk-sensitive agents as ones who have non-state-separable preferences, as in Epstein and Zin (1989), but with a value for the intertemporal elasticity of substitution equal to one (see Tallarini 2000).

where $s_0 | \mathcal{I}_0 \sim N(\widehat{s}_0, \overline{\sigma}^2)$, $\widehat{s}_t = E_t[s_t]$ is the perceived state variable, θ is the optimal weight on the new observation of the state, and ξ_{t+1} is the endogenous noise. The optimal choice of the weight θ is given by $\theta(\kappa) = 1 - 1/\exp(2\kappa) \in [0, 1]$. The following proposition summarizes the solution to the RI-RS model when $\beta R = 1$:

Proposition 3 *Given finite channel capacity κ and the degree of risk-sensitivity α , the consumption function of a risk-sensitive consumer under RI*

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (25)$$

where

$$\Pi = R\alpha\omega_\eta^2 \in (0, 1), \quad (26)$$

$$\omega_\eta^2 = \text{var}[\eta_{t+1}] = \frac{\theta}{1 - (1-\theta)R^2} \omega_\zeta^2, \quad (27)$$

η_{t+1} is defined in (10), and $\theta(\kappa) = 1 - 1/\exp(2\kappa)$.

Proof. See Appendix 7.2. ■

Comparing (20) and (25), it is straightforward to show that it is impossible to distinguish between RB and RS under RI using only consumption-savings decisions.

Proposition 4 *Let the following expression hold:*

$$\alpha = \frac{1}{2\vartheta}. \quad (28)$$

Then consumption and savings are identical in the RS-RI and RB-RI models.

Note that (28) is *exactly the same as* the observational equivalence condition obtained in the full-information RE model (see Backus, Routledge, and Zin 2004). That is, under the assumption that the agent distrusts the Kalman filter equation, the OE result obtained under full-information RE still holds under RI.¹¹

HST (1999) show that as far as the *quantity* observations on consumption and savings are concerned, the robustness version ($\vartheta > 0$ or $\alpha > 0, \widetilde{\beta}$) of the PIH model is observationally equivalent to the standard version ($\vartheta = \infty$ or $\alpha = 0, \beta = 1/R$) of the PIH model for a unique

¹¹Note that the OE becomes

$$\frac{\alpha\theta}{1 - (1-\theta)R^2} = \frac{1}{2\vartheta},$$

if we assume that the agents distrust the income process hitting the budget constraint, but trust the RI-induced noise hitting the Kalman filtering equation.

pair of discount factors.¹² The intuition is that introducing a preference for risk-sensitivity (RS) or a concern about robustness (RB) increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RS or RB on consumption and investment.¹³ Alternatively, holding all parameters constant except the pair (α, β) , the RI version of the PIH model with RB consumers ($\vartheta > 0$ and $\beta R = 1$) is observationally equivalent to the standard RI version of the model ($\vartheta = \infty$ and $\tilde{\beta} > 1/R$). To do so, we compare (7) with (16) and (25), and obtain the following OE expression for the discount factor:

Proposition 5 *Let*

$$\tilde{\beta} = \frac{1}{R} \frac{1 - R\omega_\eta^2/(2\vartheta)}{1 - R^2\omega_\eta^2/(2\vartheta)} = \frac{1}{R} \frac{1 - R\alpha\omega_\eta^2}{1 - R^2\alpha\omega_\eta^2} > \frac{1}{R}.$$

Then consumption and savings are identical in the RI, RB-RI, and RS-RI models.

3 Risk-sensitive Filtering

In Section 2.3 we solved an RS-RI model with the regular Kalman filter, and find that the OE condition between RB and RS under RI is *exactly the same as* the OE condition obtained in the full-information RE model. However, when the state cannot be observed perfectly, the classical Kalman filter that minimizes (maximizes) the expected value of a certain quadratic loss (revenue) function is no longer optimal in the risk-sensitive LQ setting. The reason is that in the risk-sensitive LQ problem the objective function is an exponential-quadratic cost function; consequently, the risk-sensitive filter is more suitable than the Kalman filter (the conditional mean estimator) for estimating the imperfectly observed state.¹⁴

3.1 Deriving the Risk-sensitive Filter Gain

In this section we will explore how the RS filtering affects consumption dynamics and precautionary savings and show that the OE between RB and RS is no longer linear, but takes a non-linear form. Note that when the state s cannot be observed perfectly, we need to know the evolution of the perceived state \hat{s} before we solve for the optimal behavior of the RS agent.

¹²HST (1999) derive the observational equivalence result by fixing all parameters, including R , except for the pair (α, β) .

¹³As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters (α, β) within the observationally-equivalent set of parameters.

¹⁴See Whittle (1990), Speyer, Fan, and Banavar (1992), and Banavar and Speyer (1998) for detailed discussions and proofs about risk-sensitive filtering.

Given the RS preference specified in Section 2.3, rather than minimizing the weighted quadratic sum of the squares of the estimation error (which gives rise to the Kalman filter) in the LQ setting, in the risk-sensitive LQ setting the agent would minimize the exponential cost criterion to obtain the risk-sensitive filter. A risk-sensitive filter solves the minimization problem

$$\min_{\{s_t\}} \mathcal{C}_t(\alpha) = \min_{\{s_t\}} E[\exp(\alpha \mathcal{J}_t)], \quad (29)$$

where

$$\mathcal{J}_t = \sum_{t=1}^N (s_t - \hat{s}_t)^T (s_t - \hat{s}_t),$$

where $\alpha > 0$ (< 0) means risk-averse (risk-tolerant), \hat{s}_t is the perceived state that is a causal function of the measurement history. The procedure used to solve (29) is provided in Speyer, Fan, and Banavar (1992). The following proposition summarizes the main results about the risk-sensitive filter in the permanent income model:

Proposition 6 *Given the constraint specified in (2), the evolution of the perceived state \hat{s}_t follows:*

$$\hat{s}_t = R\hat{s}_{t-1} - c_t + \theta_t [s_t^* - (R\hat{s}_{t-1} - c_t)],$$

where $s_t^* = s_t + \xi_t$ is the noisy signal. The RS filter gain θ can be written as

$$\theta_t = (\Psi_t^{-1} + \Lambda_t^{-1})^{-1} \Lambda_t^{-1}, \quad (30)$$

and the prior-observation variance Ψ_t is propagated according to

$$\Psi_{t+1} = R^2 (\Psi_t^{-1} + \Lambda_t^{-1} - \alpha)^{-1} + \Omega, \quad (31)$$

where $\Omega = \omega_\zeta^2$ and $\Lambda = \lambda^2 = \text{var}_t[\xi_{t+1}]$.

In our univariate RI permanent income model, the post-observation variance Σ_t is determined by channel capacity κ : $\Sigma = \sigma^2 = \frac{\Omega}{\exp(2\kappa) - R^2}$. Note that given the budget constraint, (2), the following equation always holds under RI:

$$\Psi = R^2 \Sigma + \Omega.$$

In the steady state, (31) reduces to

$$\Psi^{-1} + \Lambda^{-1} - \alpha = \Sigma^{-1},$$

which can be used to determine the value of the variance of the endogenous noise:

$$\Lambda^{-1} = \Sigma^{-1} - \Psi^{-1} + \alpha \quad (32)$$

and the risk-sensitive filter gain:

$$\begin{aligned}\theta &= (\Psi^{-1} + \Lambda^{-1})^{-1} \Lambda^{-1}, \\ &= (\Sigma^{-1} + \alpha)^{-1} (\Sigma^{-1} - \Psi_t^{-1} + \alpha) \\ &= \frac{\Sigma^{-1} - (R^2 \Sigma + \Omega)^{-1} + \alpha}{\Sigma^{-1} + \alpha}.\end{aligned}$$

To evaluate the effects of risk-sensitivity and channel capacity on the dynamics governed by the risk-sensitive filter gain θ , we rewrite the expression for θ as

$$\theta(\alpha, \kappa) = \frac{\Omega \Sigma^{-1} - (R^2 \Sigma \Omega^{-1} + 1)^{-1} + \alpha \Omega}{\Omega \Sigma^{-1} + \alpha \Omega} \quad (33)$$

$$= 1 - \frac{1}{\exp(2\kappa)} \frac{\exp(2\kappa) - R^2}{\exp(2\kappa) - R^2 + \alpha \Omega}, \quad (34)$$

which reduces to $\theta = 1 - 1/\exp(2\kappa)$ when $\alpha = 0$. Since $\alpha \Omega > 0$, it is clear that

$$\frac{\partial \theta(\alpha, \kappa)}{\partial \alpha} > 0, \quad \frac{\partial \theta(\alpha, \kappa)}{\partial \kappa} > 0, \quad (35)$$

that is, both risk-sensitivity and finite capacity increase the Kalman gain. Since the model's dynamic behavior is governed by the Kalman gain, introducing the RS filtering alters the dynamics of consumption and savings.

3.2 Consumption Function under RS and RI

Combining this RS filtering part with the RS problem proposed in Section 2.3, we can see that the RS preference affects both the dynamics of the perceived state and the functional form of optimal decision rules. The following proposition summarizes the solution to the RS model when $\beta R = 1$:

Proposition 7 *Given the risk-sensitive filter gain θ and the degree of risk-sensitivity α , the consumption function of a risk-sensitive consumer under RI*

$$c_t = \frac{R-1}{1-\Pi} \hat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (36)$$

where

$$\Pi = R\alpha\omega_\eta^2 \in (0, 1) \quad (37)$$

$$\omega_\eta^2 = \text{var}[\eta_{t+1}] = \frac{\theta(\alpha, \kappa)}{1 - (1 - \theta(\alpha, \kappa))R^2} \omega_\zeta^2, \quad (38)$$

and η_{t+1} is defined in (10).

Proof. See Appendix 7.4. ■

It is clear from (36) that the marginal propensity of consumption out of the perceived state (\widehat{s}_t) is affected by the RS preference via both the risk-sensitive control and risk-sensitive filtering. Specifically, the RS preference α affects the consumption function through Π . Second, it also affects ω_η^2 through changes in the risk-sensitive Kalman gain θ .

Comparing (20) and (36), it is straightforward to show that it is also impossible to distinguish between RB and RS under RI and RS filtering using only consumption-savings decisions.

Proposition 8 *Let the following expression hold:*

$$\vartheta \triangleq \vartheta(\alpha, \kappa, R) = \frac{1}{2} \left[\alpha \frac{1 - (1 - \theta(\kappa)) R^2}{\theta(\kappa)} \frac{\theta(\alpha, \kappa)}{1 - (1 - \theta(\alpha, \kappa)) R^2} \right]^{-1}. \quad (39)$$

Then consumption and savings are identical in the RS-RI and RB-RI models.

Proof. Equalizing (20) and (36) yields

$$\frac{1}{2\vartheta} \frac{\theta(\kappa)}{1 - (1 - \theta(\kappa)) R^2} = \alpha \frac{\theta(\alpha, \kappa)}{1 - (1 - \theta(\alpha, \kappa)) R^2},$$

where $\theta(\kappa) = 1 - 1/\exp(2\kappa)$ and $\theta(\alpha, \kappa) = 1 - \frac{1}{\exp(2\kappa)} \frac{\exp(2\kappa) - R^2}{\exp(2\kappa) - R^2 + \alpha\Omega}$. By simple calculation, we obtain (39). ■

Note that (39) means that the observational equivalence condition under RI in this RS filtering case is highly *nonlinear* and also depends on the other two model parameters: channel capacity (κ) and the interest rate (R) (or equivalently the discount factor β). Figure 1 illustrates the relationship between RS (α) and RB (ϑ) when the OE holds for different values of capacity (κ) given that $R = 1.01$ and $\Omega = \omega_\zeta^2 = 1$. Holding the degree of attention (κ) fixed, there is a nonlinear one-to-one correspondence between α and ϑ . In addition, given ϑ , the higher the degree of inattention (the smaller κ is), the higher the value of α required to maintain the OE between RS and RB.

4 Robust Kalman Filter Gain

4.1 Introducing Robustness in the Kalman Gain

Another source of robustness could arise from the Kalman filter gain. In Section 2, we assume that the agent distrusts the innovation to the perceived state but trusts the regular Kalman filter gain. Following Hansen and Sargent (Chapter 17, 2008), in this section we consider a

situation in which the agent pursues a robust Kalman gain. Specifically, assume that at t the agent observe the noisy signal

$$s_t^* = s_t + \xi_t, \quad (40)$$

where s_t is the true state and ξ_t is the iid endogenous noise and its variance, $\lambda_t^2 = \text{var}[\xi_t]$, is determined by

$$\lambda_t^2 = \lambda^2 = \text{var}[\xi_t] = \frac{(\omega_\zeta^2 + R^2\sigma^2)\sigma^2}{\omega_\zeta^2 + (R^2 - 1)\sigma^2},$$

and $\sigma^2 = \frac{\omega_\zeta^2}{\exp(2\kappa) - R^2}$ is the steady state conditional variance. Given the budget constraint,

$$s_{t+1} = R s_t - c_t + \zeta_{t+1}, \quad (41)$$

we consider the following time-invariant robust Kalman filter equation,

$$\widehat{s}_{t+1} = (1 - \theta)(R\widehat{s}_t - c_t) + \theta(s_{t+1} + \xi_{t+1}), \quad (42)$$

where \widehat{s}_{t+1} is the estimate of the state using the history of the noisy signals, $\{s_j^*\}_{j=0}^{t+1}$. We want θ to be robust to possible misspecification of Equations (40) and (41). To obtain robust Kalman filter gain, the agent considers the following distorted model:

$$s_{t+1} = R s_t - c_t + \zeta_{t+1} + \omega \nu_{1,t+1}, \quad (43)$$

$$s_{t+1}^* = s_{t+1} + \xi_{t+1} + \varrho \nu_{2,t+1}, \quad (44)$$

where $\nu_{t+1} = \begin{bmatrix} \nu_{1,t+1} & \nu_{2,t+1} \end{bmatrix}^T$ are distortions to the conditional means of the two shocks, ζ_{t+1} and ξ_{t+1} .

Combining (43) with (44) gives the following dynamic equation for the estimation error:¹⁵

$$e_{t+1} = (1 - \theta) R e_t + (1 - \theta) \zeta_{t+1} - \theta \xi_{t+1} + (1 - \theta) \omega \nu_{1,t+1} - \theta \varrho \nu_{2,t+1}. \quad (45)$$

As shown in Hansen and Sargent (2007), we can solve for the robust Kalman filter gain corresponding to this problem by solving the following deterministic optimal linear regulator:

$$v(e_0) = e_0^T P e_0 = \max_{\{\nu_{t+1}\}} \sum_{t=0}^{\infty} (e_t^T e_t - \vartheta \nu_{t+1}^T \nu_{t+1}), \quad (46)$$

subject to

$$e_{t+1} = (1 - \theta) R e_t + D \nu_{t+1}, \quad (47)$$

¹⁵Note that control variable, c , does not affect the estimation error equation.

where $D = \begin{bmatrix} (1 - \theta)\omega & -\theta\varrho \end{bmatrix}$. We can compute the worst-case shock by solving the corresponding Bellman equation

$$\nu_{t+1}^* = Qe_t, \quad (48)$$

where

$$Q = (\vartheta I - D^T P D)^{-1} D^T P (1 - \theta) R. \quad (49)$$

Note that Q is a function of robustness (ϑ) and channel capacity (κ).

For arbitrary Kalman filter gain θ and (48), the error in reconstructing the state s can be written as

$$e_{t+1} = \{(1 - \theta) R + [(1 - \theta)\omega - \theta\varrho] Q\} e_t + (1 - \theta) \zeta_{t+1} - \theta \xi_{t+1}. \quad (50)$$

Taking unconditional mean on both sides of (50) gives

$$\Sigma_{t+1} = \{(1 - \theta) R + [(1 - \theta)\omega - \theta\varrho] Q\} \Sigma_t + (1 - \theta)^2 \omega_\zeta^2 + \theta^2 \omega_\xi^2, \quad (51)$$

where $\Sigma_{t+1} = E[e_{t+1}^2]$. From (51), it follows directly that in the steady state

$$\Sigma(\theta; Q) = \frac{(1 - \theta)^2 \omega_\zeta^2 + \theta^2 \omega_\xi^2}{1 - \chi^2},$$

where $\chi = (1 - \theta) R + [(1 - \theta)\omega - \theta\varrho] Q$. We use the program `rfilter.m` provided in Hansen and Sargent (2008) to compute the robust Kalman filter gain $\theta(\vartheta, \kappa)$ that minimizes the variance of e_t , $\Sigma(\theta; Q)$:

$$\theta(\vartheta, \kappa) = \arg \min (\Sigma(\theta; Q(\vartheta, \kappa))). \quad (52)$$

Figure 2 illustrates how robustness (measured by ϑ) and inattention (measured by κ) affect the robust Kalman gain when $R = 1.01$ and $\omega_\zeta = 1$. It clearly shows that holding the degree of attention (i.e., channel capacity κ) fixed, increasing robustness (i.e., reducing ϑ) can increase the Kalman gain (θ). In addition, for given robustness (ϑ), the Kalman gain is increasing with capacity. For example, when $\log(\vartheta) = 3$, the robust Kalman gain will increase from 60.17% to 77.35% when capacity κ increases from 0.6 bits to 1 bit;¹⁶ when $\kappa = 0.6$ bits, the robust Kalman gain will increase from 58.31% to 60.17% when ϑ reduces from $\log(\vartheta) = 4$ to 3.¹⁷

4.2 Consumption Function and Comparison with RS Filtering

After obtaining the robust Kalman gain $\theta(\vartheta, \kappa)$, we can solve the Bellman equation proposed in Section 2.2 using the Kalman filtering equation with robust θ and obtain a new consumption

¹⁶ θ measures how much uncertainty can be removed upon receiving new signals on the state.

¹⁷This result is consistent with that obtained in a continuous-time setting discussed in Kasa (2003).

function under RB and RI:

$$c_t = \frac{R-1}{1-\Pi} \widehat{s}_t - \frac{\Pi \bar{c}}{1-\Pi}, \quad (53)$$

where

$$\begin{aligned} \Pi &= \frac{R\omega_\eta^2}{2\vartheta_0} \in (0, 1), \\ \omega_\eta^2 &= \text{var}[\eta_{t+1}] = \frac{\theta(\vartheta, \kappa)}{1 - (1 - \theta(\vartheta, \kappa))R^2} \omega_\zeta^2, \end{aligned} \quad (54)$$

and \widehat{s}_t is governed by

$$\widehat{s}_{t+1} = \rho_s \widehat{s}_t + \eta_{t+1}, \quad (55)$$

where $\rho_s = \frac{1-R\Pi}{1-\Pi} \in (0, 1)$.

Note that here θ is a function of both ϑ (concerns about Kalman gain) and κ (channel capacity), rather than simply $1 - 1/\exp(2\kappa)$ as obtained in Section 2.2. In this case the agent has two types of concerns about model misspecification: (i) concerns about the disturbances to the perceived permanent income (ϑ_0) and (ii) concerns about the Kalman gain (ϑ). It is clear from (53) and (54) that the two types of robustness have opposing effects on both the marginal propensity to consume out of permanent income, i.e., the responsiveness of c_t to \widehat{s}_t (MPC = $\frac{R-1}{1-\Pi}$) and precautionary savings, i.e., the intercept of the consumption profile (PS = $\frac{\Pi \bar{c}}{1-\Pi} = -\bar{c} + \frac{\bar{c}}{1-\Pi}$).¹⁸ Specifically, the less the value of ϑ_0 (Type I robustness) the larger the MPC and the larger the precautionary saving increment, since

$$\frac{\partial(\text{MPC})}{\partial\vartheta_0} < 0 \text{ and } \frac{\partial(\text{PS})}{\partial\vartheta_0} < 0.$$

For the effects of Type II robustness (ϑ), the less the value of ϑ the less the MPC and the less the precautionary saving increment

$$\frac{\partial(\text{MPC})}{\partial\vartheta} > 0 \text{ and } \frac{\partial(\text{PS})}{\partial\vartheta} > 0$$

because $\frac{\partial\omega_\eta^2}{\partial\theta} < 0$, $\frac{\partial\theta}{\partial\vartheta} < 0$.

From (53) and (54), it is clear that the precautionary savings increment in the RB-RI model is determined by the interaction of three factors: labor income uncertainty, preferences for robustness (RB), and finite information-processing capacity (RI). The intuition about the effects of Type I robustness (ϑ_0) on precautionary savings is as follows. Since agents with low capacity are very concerned about the confluence of low permanent income and high consumption (meaning they believe their permanent income is high so they consume a lot and then their new signal

¹⁸Note that given the consumption function Π has the same impact on the marginal propensity to consume and precautionary savings.

indicates that in fact their permanent income was low), they take actions which reduce the probability of this bad event – they save more. The strength of the precautionary effect is positively related to the amount of uncertainty regarding the true level of permanent income, and this uncertainty increases as θ gets smaller. The intuition about the effects of Type II robustness (ϑ) on precautionary savings is as follows. An increase in Type II robustness (a reduction in ϑ) will increase the Kalman gain θ , which leads to lower ω_η^2 and then low precautionary savings. Figure 3 illustrates how Type II robustness (ϑ) and channel capacity (κ) affect ω_η^2 . In addition, since

$$\frac{\partial \omega_\eta^2}{\partial \vartheta} > 0, \quad \frac{\partial \omega_\eta^2}{\partial \kappa} < 0, \quad \frac{\partial \omega_\eta^2}{\partial \vartheta_0} = 0$$

it is clear that under certain conditions a greater reaction to the shock can either be interpreted as an increased concern for robustness in the presence of model misspecification, or an increase in information-processing ability when agents only have finite channel capacity. Figure 4 illustrates Π as functions of ϑ_0 and ϑ . It clearly shows that how increasing the robustness preference for the shock to the perceived state (i.e., decreasing ϑ_0) and reducing the preference for a robust gain (i.e., increasing ϑ) increases Π and then increase the impacts of the two types of robustness on consumption and precautionary savings.

Comparing (36) with (53), it is clear that it is also impossible to distinguish between RB and RS under RI and RS filtering using only consumption-savings decisions.

Proposition 9 *Let the following expression hold:*

$$\alpha \frac{\theta(\alpha, \kappa)}{1 - (1 - \theta(\alpha, \kappa)) R^2} = \frac{1}{2\vartheta_0} \frac{\theta(\vartheta, \kappa)}{1 - (1 - \theta(\vartheta, \kappa)) R^2}, \quad (56)$$

where $\theta(\alpha, \kappa) = 1 - \frac{1}{\exp(2\kappa)} \frac{\exp(2\kappa) - R^2}{\exp(2\kappa) - R^2 + \alpha\Omega}$ and $\theta(\vartheta, \kappa)$ is determined by (52). Then consumption and savings are identical in the RS-RI and RB-RI models.

Here we assume that the values of ϑ_0 and ϑ are the same, that is, one parameter governs the two types of robustness. It is straightforward to show that there is one-to-one correspondence between α and ϑ (or ϑ_0) because Π is a strictly increasing function of α or ϑ . In other words, (56) is another OE condition between RB and RS. It is also highly *nonlinear* and also depends on channel capacity (κ) and the interest rate (R). Figure 5 illustrates the existence of OE in this case with robust Kalman and risk-sensitive Kalman when $R = 1.01$ and $\omega_\zeta^2 = 1$. It clearly shows that there does exist an OE between RB and RS in this model. Note that if ϑ_0 and ϑ are not the same, we can find more OE combinations as we have an additional free parameter.

5 Implications for Consumption Dynamics, Aggregate Savings and Welfare

In this section we examine the implications of robust or risk-sensitive Kalman filtering for consumption dynamics and welfare losses due to finite capacity. For simplicity, we focus on the OE cases when comparing welfare losses of RS (or RB) agents with agents with higher discount factors.

5.1 Sensitivity and Smoothness of Consumption Process

Combining (53) with (55) yields the change in individual consumption in the RI-RB economy:

$$\Delta c_t = \frac{(1-R)\Pi}{1-\Pi} (c_{t-1} - \bar{c}) + \frac{R-1}{1-\Pi} \left[\frac{\theta \zeta_t}{1 - (1-\theta)R \cdot L} + \theta \left(\xi_t - \frac{\theta R \xi_{t-1}}{1 - (1-\theta)R \cdot L} \right) \right], \quad (57)$$

where L is the lag operator and we assume that $(1-\theta)R < 1$.¹⁹ This expression shows that consumption growth is a weighted average of all past permanent income and noise shocks. In addition, it is also clear from (57) that the propagation mechanism of the model is determined by the robust Kalman filter gain, $\theta(\vartheta, \kappa)$. Figure ?? illustrates that consumption in the RB-RI model reacts gradually to income shocks, with monotone adjustments to the corresponding RB asymptote. Note that when $\vartheta_0 = 1$ and $\log(\vartheta) = 3$, the robust Kalman gain $\theta = 42.56\%$. This case is illustrated by the dash-dotted line in Figure 6. Similarly, the dotted line corresponds to the case in which $\vartheta_0 = 2$ and $\log(\vartheta) = 5$ ($\theta = 0.3541$). With a stronger preference for robustness, the precautionary savings increment is larger and thus an income shock that is initially undetected would have larger impacts on consumption during the adjustment process.

Using (57), the relative volatility of consumption growth relative to income growth can be written as

$$\mu = \frac{\text{sd}[\Delta c_t]}{\text{sd}[\Delta y_t]} = \frac{\theta}{1-\Pi} \sqrt{\sum_{j=0}^{\infty} \Upsilon_j^2 + \frac{1-\theta}{\theta(1-(1-\theta)R^2)} \sum_{j=0}^{\infty} (\Upsilon_j - R\Upsilon_{j-1})^2}, \quad (58)$$

where we use the fact that $\omega_{\xi}^2 = \text{var}[\xi_t] = \frac{1-\theta}{\theta(1-(1-\theta)R^2)} \omega_{\zeta}^2$, $\rho_1 = \frac{1-R\Pi}{1-\Pi} \in (0, 1)$, $\rho_2 = (1-\theta)R \in (0, 1)$, and

$$\Upsilon_j = \sum_{k=0}^j \left(\rho_1^{j-k} \rho_2^k \right) - \sum_{k=0}^{j-1} \left(\rho_1^{j-1-k} \rho_2^k \right), \text{ for } j \geq 1,$$

¹⁹This assumption requires $\kappa > \frac{1}{2} \log(R) \approx \frac{R-1}{2}$, which is weaker than the condition needed for convergence of the filter.

and $\Upsilon_0 = 1$. Figure 7 illustrates how the combination of the two types of robustness, ϑ_0 and ϑ , affects the relative volatility of consumption growth to income growth when $\kappa = 0.3$ bits. It clearly shows that given ϑ_0 , the relative volatility μ is increasing with ϑ . The intuition is that reducing ϑ (i.e., increasing Type I robustness) increases the robust Kalman gain θ and reduces ω_η^2 and Π , which leads to more smooth consumption process. Note that θ is independent of ϑ_0 . Hence, given ϑ , μ is decreasing with ϑ_0 because Π is increasing with ϑ_0 . To explore the intuition behind this result, we consider the perfect-state-observation case in which $\kappa = \infty$. In this case, the relative volatility of consumption growth to income growth reduces to

$$\mu = \frac{1}{1 - \Pi} \sqrt{\frac{2}{1 + \rho_1}}, \quad (59)$$

which clearly shows that ϑ_0 increases the relative volatility via two channels. First, a higher ϑ_0 increases the marginal propensity to consume out of permanent income $\left(\frac{R-1}{1-\Pi}\right)$, and second, it increases consumption volatility by reducing the persistence of permanent income measured by ρ_1 : $\frac{\partial \rho_1}{\partial \Pi} < 0$.

In the presence of robustness, rational inattention measured by κ affects consumption volatility via two channels: (i) the gradual and smooth responses to income shocks (i.e., the $1 - (1 - \theta)R \cdot L$ term in (57) and (ii) the RI-induced noises (ξ_t). Specifically, a reduction in capacity κ decreases the Kalman gain θ , which strengthens the smooth responses to income shock and increases the volatility of the RI-induced noise. Luo (2008) shows that the noise effect dominates the smooth response effect, and the volatility of consumption growth decreases with κ . Figure 8 illustrates how the combination of ϑ_0 and ϑ affects the relative volatility of consumption growth to income growth when $\kappa = 0.3$ bits and there is no noise term. In this case,

$$\Delta c_t = \frac{(1 - R) \Pi}{1 - \Pi} (c_{t-1} - \bar{c}) + \frac{R - 1}{1 - \Pi} \frac{\theta \zeta_t}{1 - (1 - \theta)R \cdot L}.$$

Figure 8 shows that given ϑ_0 , the relative volatility μ is decreasing with ϑ . The intuition is that reducing ϑ (i.e., increasing Type I robustness) increases the robust Kalman gain θ , which leads to more volatile consumption process because the smooth response effect completely dominates the noise effect.

5.2 Implications of RB and RI on Aggregate Savings

In this section, following Caballero (1994), Irene and Wang (1994), and Wang (2000), we consider an aggregate economy with a continuum of risk averse agents. In this aggregate economy, we examine how two types of induced uncertainty, model uncertainty due to RB and state uncertainty due to RI, affect aggregate savings. Specifically, we assume that p is the probability

of surviving through period $t + 1$, given that one is alive at t . The unconditional probability of reaching age t is p^{t-1} , and the effective discount factor can be written as $p\beta$.

Combining (57) with the original budget constraint, $b_{t+1} = Rb_t + y_t - c_t$, and the consumption function (53), we obtain the evolution of individual financial wealth:

$$b_{t+1} = \rho_s b_t + \frac{\rho_s - \rho}{R - \rho} y_t + \frac{\Pi \bar{c}}{1 - \Pi} + \zeta_{t+1} \quad (60)$$

where

$$\zeta_{t+1} = (\theta - 1) \left(\zeta_{t+1} - \frac{\theta R \zeta_t}{1 - (1 - \theta) R \cdot L} \right) + \theta \left(\xi_{t+1} - \frac{\theta R \xi_t}{1 - (1 - \theta) R \cdot L} \right), \quad (61)$$

which reduces to 0 as there is no capacity constraint ($\kappa = \infty$ and $\xi = 0$). (60) can be written recursively to yield

$$b_{t+1} = \rho_s^{t+1} b_0 + \frac{\rho_s - \rho}{R - \rho} \sum_{j=0}^t \rho_s^{t-j} y_j + \sum_{j=0}^{t+1} \rho_s^{t+1-j} \zeta_j + \frac{\Pi \bar{c}}{1 - \Pi} \frac{1 - \rho_s^{t+1}}{1 - \rho_s}. \quad (62)$$

Assume that the initial cross-sectional distribution of the income shock is its stationary distribution. Given that there are a continuum of agents in the model economy, the law of large numbers (LLN) holds and we can thus construct the space of agents and the probability space appropriately such that aggregate income is constant and all idiosyncratic shocks are cancelled out. As the population size of the age t group is $(1 - p)p^{t-1}$, aggregate wealth in the economy, A , can be written as

$$A = (1 - p) \sum_{t=1}^{\infty} \left(p^{t-1} \tilde{b}_t \right), \quad (63)$$

where $\tilde{b}_t = \rho_s^t b_0 + \frac{\rho_s - \rho}{R - \rho} \sum_{j=0}^{t-1} \rho_s^{t-1-j} y_j + \frac{\Pi \bar{c}}{1 - \Pi} \frac{1 - \rho_s^t}{1 - \rho_s}$. At any period, there are $1 - p$ newborns entering the economy and demanding a total endowment, $(1 - p) b_0$. At the same time, $1 - p$ individuals die, leaving a total accidental wealth, $(1 - p) A$. Following Caballero (1994), the equilibrium condition is that the total supply of wealth equals the total demand for wealth:

$$b_0 = A. \quad (64)$$

Combining (63) with (64), we can obtain the equilibrium aggregate wealth

$$A^* = (1 - p) \frac{\rho_s - \rho}{R - \rho} \frac{1 - p \rho_s}{1 - \rho_s} \sum_{t=1}^{\infty} \sum_{j=0}^{t-1} \left(p^{t-1} \rho_s^{t-1-j} y_j \right) + \frac{1}{1 - p \rho_s} \frac{\Pi \bar{c}}{1 - \Pi}. \quad (65)$$

The steady state equilibrium implies that

$$y_t = \bar{y}$$

for any t . In this case, (65) can be rewritten as

$$A^* = \underbrace{\frac{\rho_s - \rho}{R - \rho} \frac{1}{1 - \rho_s} \frac{1 - p}{p} \bar{y}}_{A_N} + \underbrace{\frac{1}{1 - \rho_s} \frac{\Pi \bar{c}}{1 - \Pi}}_{A_P}, \quad (66)$$

where A_N and A_P are the non-precautionary and precautionary components, respectively. Denote μ the relative importance of precautionary saving to non-precautionary wealth

$$\mu = \frac{A_P}{A_N} = \frac{p(R - \rho)}{(1 - p)(\rho_s - \rho)} \frac{\Pi}{1 - \Pi} \frac{\bar{c}}{\bar{y}}.$$

Note that here the expression of A_N is not valid in the full-information case in which $\rho_s = 1$ as it is not well defined in this case. Since $\rho_s = \frac{1 - R\Pi}{1 - \Pi}$ and Π depends on both RB and RI, we have the following proposition.

Proposition 10 *In the steady state equilibrium, RB and RI not only affects the level of aggregate precautionary savings (A_P), but also affects the level of non-precautionary wealth (A_N). Specifically, RB and RI reduce the level of non-precautionary wealth and increases the level of precautionary savings; as a result, they increase the relative importance of precautionary savings in aggregate wealth.*

5.3 Welfare Effects of RB and RI

In this section, we examine how the interaction of the two types of robustness and inattention affect consumer welfare, and compare different welfare implications in the RB-RI and RI settings. To make the comparisons meaningful, we restrict our attention to combinations of preferences that imply observational equivalence – that is, how agents whose consumption/savings decisions are identical (but for different reasons) suffer different welfare losses from information-processing capacity limitations.

Specifically, in the RB-RI model, the value function can be written as:

$$\begin{aligned} \hat{v}^{RB-RI}(\hat{s}_t) = & -\frac{\beta R^2 - 1}{2(\beta - \omega_\eta^2/(2\vartheta))} \hat{s}_t^2 + \frac{(\beta R^2 - 1)\bar{c}}{(R - 1)(\beta - \omega_\eta^2/(2\vartheta))} \hat{s}_t \\ & - \left[\frac{1}{2(R - 1)^2(\beta - \omega_\eta^2/(2\vartheta))} \bar{c}^2 - \frac{\beta}{1 - \beta} \vartheta \log \left(1 - \frac{\beta R^2 - 1}{\beta R - R\omega_\eta^2/(2\vartheta)} \frac{R\omega_\eta^2}{2\vartheta} \right) \right], \end{aligned} \quad (67)$$

which reduces to

$$\begin{aligned} \hat{v}^{RB-RI}(\hat{s}_t) = & -\frac{R - 1}{2(1 - R\omega_\eta^2/(2\vartheta))} \hat{s}_t^2 + \frac{R\bar{c}}{1 - R\omega_\eta^2/(2\vartheta)} \hat{s}_t \\ & - \left[\frac{1}{2(R - 1)(1 - R\omega_\eta^2/(2\vartheta))} \bar{c}^2 - \frac{1}{R - 1} \vartheta \log \left(1 - \frac{R - 1}{1 - R\omega_\eta^2/(2\vartheta)} \frac{R\omega_\eta^2}{2\vartheta} \right) \right], \end{aligned} \quad (68)$$

when $\beta R = 1$. This value functions can be used to compute the *marginal* welfare losses due to RI at various levels of (i) channel capacities (κ), (ii) Type I robustness (ϑ_0), and (iii) Type II robustness (ϑ). Specifically, following Barro (2007) and Luo and Young (2010), the marginal welfare costs (mwc) due to finite capacity in the RB-RI model can be written as

$$\begin{aligned} \text{mwc}^{RB-RI} &= \frac{\partial \widehat{v}^{RB-RI} / \partial \kappa}{(\partial \widehat{v}^{RB-RI} / \partial \widehat{s}_0) \widehat{s}_0} \\ &= -\frac{1}{2} \frac{\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}} \frac{R-1}{1-(R-1)\omega_\eta^2/(2\vartheta)} + \frac{R-1}{(1-\beta^{RB-RI})^2} \log\left(1 - (R-1) \frac{\omega_\eta^2}{2\vartheta_0}\right)}{\frac{R(R-1)}{R-R\omega_\eta^2/(2\vartheta)} \widehat{s}_0^2 + \frac{R\bar{c}}{1-R\omega_\eta^2/(2\vartheta)} \widehat{s}_0} \frac{\partial \omega_\eta^2}{\partial \theta} \frac{\partial \theta}{\partial \kappa} \end{aligned}$$

respectively. Note that here $\partial \widehat{v}^{RB-RI} / \partial \kappa$ and $\partial \widehat{v}^{RB-RI} / \partial \theta$ are evaluated for given \widehat{s}_0 , and the welfare costs due to RI are compared with that from a small proportional change in the initial level of the perceived state \widehat{s}_0 .

and the corresponding value function for the standard RI model is

$$\widehat{v}(\widehat{s}_t) = -\frac{\beta R^2 - 1}{2\beta} \widehat{s}_t^2 + \frac{(\beta R^2 - 1) \bar{c}}{\beta(R-1)} \widehat{s}_t - \frac{1}{2} \frac{\beta R^2 - 1}{\beta(R-1)^2} \bar{c}^2 - \frac{\beta R^2 - 1}{2(1-\beta)} \omega_\eta^2. \quad (69)$$

In the RI case, when the standard restriction on the discount factor β , $\beta R = 1$, holds, the value function can be rewritten as

$$\widehat{v}(\widehat{s}_t) = -\frac{(R-1)R}{2} \widehat{s}_t^2 + R\bar{c}\widehat{s}_t - \frac{R}{2(R-1)} \bar{c}^2 - \frac{R}{2} \omega_\eta^2. \quad (70)$$

It is straightforward to show that when the discount factor β in Expression (67) satisfies

$$\beta = \beta^{RB-RI} = \frac{1}{R} - (R-1) \frac{\omega_\eta^2(\kappa, \vartheta)}{2\vartheta_0}, \quad (71)$$

the observational equivalence between the RB-RI model and the RI model with $\beta R = 1$ holds in the sense that they lead to the same consumption and savings rules. Given (71), (67) can be rewritten as

$$\widehat{v}^{RB-RI}(\widehat{s}_t) = -\frac{(R-1)R}{2} \widehat{s}_t^2 + R\bar{c}\widehat{s}_t - \frac{R}{2(R-1)} \bar{c}^2 - \frac{\beta^{RB-RI}}{1-\beta^{RB-RI}} \vartheta_0 \log\left(1 - (R-1) \frac{R\omega_\eta^2}{2\vartheta_0}\right) \quad (72)$$

The only difference between the two functions, (70) and (72), is the constant term involving the volatility ω_η^2 ; this equivalence arises because the slope and curvature parameters are pinned down by the equalization of the consumption decisions. We note that

$$\lim_{\vartheta \rightarrow \infty} \left\{ -\vartheta \log\left(1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta}\right) \right\} = \frac{(R-1)R}{2} \omega_\eta^2,$$

so that the two expressions are equal if $\vartheta = \infty$ and $\beta^{RB-RI} = 1/R$. We also note that

$$\frac{\partial}{\partial \vartheta} \left(-\vartheta \log \left(1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta} \right) \right) < 0.$$

Thus, we have

$$-\vartheta \log \left(1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta} \right) > \frac{(R-1) R}{2} \omega_\eta^2,$$

for finite $\vartheta > 0$. However, this result does not mean that RB-RI households have lower lifetime utility than RI agents do, conditional on being observationally equivalent, because $\beta^{RB-RI} = 1/R - (R-1) \omega_\eta^2(\kappa, \vartheta) / (2\vartheta_0) < 1/R$ and $\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}}$ in (72) is increasing with β^{RB-RI} . In other words, under the OE, i.e., given the same levels of consumption and precautionary savings, the welfare costs of uncertainty of RB agents are affected by two channels: the log term, $-\vartheta_0 \log \left(1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta_0} \right)$, and the discount factor term, $\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}}$. To analyze the effects of finite capacity and the two types of robustness on the welfare costs of uncertainty, denote

$$\Gamma = -\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}} \vartheta_0 \log(\beta^{RB-RI} R), \quad (73)$$

since $\log \left(1 - (R-1) R \frac{\omega_\eta^2}{2\vartheta_0} \right) = \log(\beta^{RB-RI} R)$. It is straightforward to show that

$$\begin{aligned} \frac{\partial \Gamma}{\partial \kappa} &= \left[-\vartheta_0 \frac{1}{1-\beta^{RB-RI}} - \vartheta_0 \log(\beta^{RB-RI} R) \frac{1}{(1-\beta^{RB-RI})^2} \right] \frac{\partial \beta^{RB-RI}}{\partial \kappa} \\ &= -\vartheta_0 \frac{1-\beta^{RB-RI} + \log(\beta^{RB-RI} R)}{(1-\beta^{RB-RI})^2} \underbrace{\frac{\partial \beta^{RB-RI}}{\partial \omega_\eta^2}}_{<0} \underbrace{\frac{\partial \omega_\eta^2}{\partial \kappa}}_{<0} < 0 \end{aligned}$$

if $1 - \beta^{RB-RI} + \log(\beta^{RB-RI} R) > 0$. Note that given that $R = 1.02$ and $\beta^{RB-RI} \in [0.9, 0.99]$, $1 - \beta^{RB-RI} + \log(\beta^{RB-RI} R) > 0$ always holds. That is, for reasonable discount factors, the welfare loss is decreasing with finite capacity, κ . This result is similar to that obtained in Luo (2008) and Luo and Young (2009). Similarly, we have

$$\frac{\partial \Gamma}{\partial \vartheta} = -\vartheta_0 \frac{1-\beta^{RB-RI} + \log(\beta^{RB-RI} R)}{(1-\beta^{RB-RI})^2} \underbrace{\frac{\partial \beta^{RB-RI}}{\partial \omega_\eta^2}}_{<0} \underbrace{\frac{\partial \omega_\eta^2}{\partial \vartheta}}_{>0} > 0$$

if $1 - \beta^{RB-RI} + \log(\beta^{RB-RI} R) > 0$. That is, for reasonable discount factors, the welfare loss is decreasing with Type II robustness (i.e., a reduction in ϑ), the concern about misspecifying the Kalman gain. The intuition is simple: the smaller the value of ϑ (i.e., the stronger the preference for robustness in the Kalman gain), the larger the value of θ and the smaller the amount of the

total uncertainty facing the consumer, ω_η^2 . Figure 9 illustrates how ϑ and κ affect the welfare loss measured by Γ given $\vartheta_0 = 1$ and $R = 1.02$. It clearly shows that Γ is increasing with ϑ and decreasing with κ .

Given the complexity of the expression for $\frac{\partial \Gamma}{\partial \vartheta_0}$ that measures the effects of Type I robustness for the welfare costs,

$$\frac{\partial \Gamma}{\partial \vartheta_0} = -\frac{1 - \beta^{RB-RI} + \log(\beta^{RB-RI} R)}{(1 - \beta^{RB-RI})^2} \left(\frac{1}{R} - \beta^{RB-RI} \right) - \frac{\beta^{RB-RI}}{1 - \beta^{RB-RI}} \log(\beta^{RB-RI} R),$$

we cannot obtain an explicit result about how ϑ_0 affects Γ via interacting with κ and ϑ . We thus use a figure to illustrate how ϑ_0 affects the welfare costs Γ . Figure 10 illustrates the effects of ϑ_0 and ϑ on Γ when $R = 1.02$ and $\kappa = 0.3$ bits.²⁰ It is clear from the figure that given κ , the welfare loss is decreasing with Type I robustness (i.e., a reduction in ϑ_0), the concern about misspecifying the Kalman filtering equation hitting the income shock and the endogenous noise. The intuition for this result is that under the OE, i.e., given the same levels of consumption and precautionary savings, ϑ_0 affects the welfare costs of uncertainty via two channels: the log term, $-\vartheta_0 \log\left(1 - (R-1)R\frac{\omega_\eta^2}{2\vartheta_0}\right)$, and the discount factor term, $\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}}$, in (72).²¹ The log term is increasing with the degree of Type I robustness (i.e., a reduction in ϑ_0), while the discount factor term is decreasing with Type I robustness because β^{RB-RI} is decreasing with Type I robustness; and the effect of ϑ_0 on the discount factor term dominates that of ϑ_0 on the log term. This result implies that the costs of uncertainty are smaller for RB-RI agents with stronger preference for Type I robustness than RI agents whenever the parameters are such that their observable behavior is identical.

Furthermore, we can use the two value functions, (70) and (72), to compute the *marginal* welfare losses due to RI at different levels of (i) channel capacities (κ), (ii) Type I robustness (ϑ_0), and (iii) Type II robustness (ϑ). Specifically, following Barro (2007) and Luo and Young (2010), the marginal welfare costs (mwc) due to finite capacity in the RB-RI model and the RI model can be written as

$$\begin{aligned} \text{mwc}^{RB-RI} &= \frac{\partial \widehat{v}^{RB-RI} / \partial \kappa}{(\partial \widehat{v}^{RB-RI} / \partial \widehat{s}_0) \widehat{s}_0} \\ &= \frac{-\frac{1}{2} \left[\frac{\beta^{RB-RI}}{1-\beta^{RB-RI}} \frac{R-1}{1-(R-1)\omega_\eta^2/(2\vartheta)} + \frac{R-1}{(1-\beta^{RB-RI})^2} \log\left(1 - (R-1)\frac{\omega_\eta^2}{2\vartheta_0}\right) \right] \frac{\partial \omega_\eta^2}{\partial \theta} \frac{\partial \theta}{\partial \kappa}}{-(R-1)R\widehat{s}_0^2 + R\widehat{c}\widehat{s}_0} \\ \text{mwc}^{RI} &= \frac{\partial \widehat{v} / \partial \kappa}{(\partial \widehat{v} / \partial \widehat{s}_0) \widehat{s}_0} = \frac{-\frac{R}{2} \frac{\partial \omega_\eta^2}{\partial \theta} \frac{\partial \theta}{\partial \kappa}}{-(R-1)R\widehat{s}_0^2 + R\widehat{c}\widehat{s}_0} \end{aligned}$$

²⁰The result is robust to other values of channel capacity κ .

²¹Note that ϑ_0 has no effect on the total uncertainty ω_η^2 .

respectively. Note that here $\partial \widehat{v}^{RB-RI} / \partial \kappa$ and $\partial \widehat{v}^{RB-RI} / \partial \theta$ are evaluated for given \widehat{s}_0 , and the welfare costs due to RI are compared with that from a small proportional change in the initial level of the perceived state \widehat{s}_0 . Therefore, the following ratio, π , measures the *relative* marginal welfare losses due to RI in the two economies at various capacities (κ):

$$\pi = \frac{\text{mwc}^{RB-RI}}{\text{mwc}^{RI}} = \frac{R-1}{1-\beta^{RB-RI}} \left(\frac{1}{R} + \frac{\log(\beta^{RB-RI} R)}{1-\beta^{RB-RI}} \right). \quad (74)$$

Expression (74) clearly shows that this ratio is different at various capacities and is determined by β^{RB-RI} that is a function of κ , ϑ , ϑ_0 , and labor income uncertainty given R . Figures 11 and 12 clearly show that the relative marginal welfare losses of RI-RB agents to RI agents are decreasing with Type I robustness (i.e., decreasing with ϑ_0), increasing with Type II robustness (i.e., decreasing with ϑ), and increasing with κ .

6 Conclusion

This paper has provided a characterization of the consumption-savings behavior of a single agent who has a preference for robustness (worries about model misspecification, RB) and the risk-sensitivity preference (RS) with limited information-processing ability. Specifically, we show that once allowing RS consumers to use the risk-sensitive filter and RB consumers to use the robust Kalman filter to update the perceived state, the absolute and linear OE between RB and RS obtained in Hansen and Sargent (2007) and Luo and Young (2010) no longer holds; instead, we find a conditional and nonlinear OE between RB and RS. Next, we find that in the filtering problem, either a stronger preference for robustness in the Kalman gain, a stronger risk-sensitive preference, or higher channel capacity can increase the Kalman gain. Finally, we examine how two types of concerns about model misspecification – (i) concerns about the disturbances to the perceived permanent income and (ii) concerns about the Kalman gain – interact with finite capacity to affect the optimal choices of consumption and savings in an otherwise standard permanent income model.

7 Appendix

7.1 Solving the Robust-RI Model with Concerns about the Income Shock

To solve the Bellman equation (12) subject to (15), we conjecture that

$$v(\widehat{s}_t) = -C - B\widehat{s}_t - A\widehat{s}_t^2, \quad (75)$$

where A , B , and C are constants to be determined. Substituting this guessed value function into the Bellman equation (12) gives

$$-C - B\widehat{s}_t - A\widehat{s}_t^2 = \max_{c_t} \min_{\nu_t} \left\{ -\frac{1}{2} (c_t - \bar{c})^2 + \beta E_t [\vartheta \nu_t^2 - C - B\widehat{s}_{t+1} - A\widehat{s}_{t+1}^2] \right\}, \quad (76)$$

subject to (15):

$$\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\zeta \nu_t + \eta_{t+1}.$$

We can do the min and max operations in any order, so we choose to do the minimization first. The first-order condition for ν_t is

$$2\vartheta \nu_t - 2AE_t [\omega_\zeta \nu_t + R\widehat{s}_t - c_t] \omega_\zeta - B\omega_\zeta = 0,$$

which means that

$$\nu_t = \frac{B + 2A(R\widehat{s}_t - c_t)}{2(\vartheta - A\omega_\zeta^2)} \omega_\zeta. \quad (77)$$

Substituting (77) back into (76) gives

$$-A\widehat{s}_t^2 - B\widehat{s}_t - C = \max_{c_t} \left\{ -\frac{1}{2} (\bar{c} - c_t)^2 + \beta E_t \left[\vartheta \left[\frac{B + 2A(R\widehat{s}_t - c_t)}{2(\vartheta - A\omega_\zeta^2)} \omega_\zeta \right]^2 - A\widehat{s}_{t+1}^2 - B\widehat{s}_{t+1} - C \right] \right\},$$

where $\widehat{s}_{t+1} = R\widehat{s}_t - c_t + \omega_\zeta \nu_t + \eta_{t+1}$. The first-order condition for c_t is

$$(\bar{c} - c_t) - 2\beta\vartheta \frac{A\omega_\zeta}{\vartheta - A\omega_\zeta^2} \nu_t + 2\beta A \left(1 + \frac{A\omega_\zeta^2}{\vartheta - A\omega_\zeta^2} \right) (R\widehat{s}_t - c_t + \omega_\zeta \nu_t) + \beta B \left(1 + \frac{A\omega_\zeta^2}{\vartheta - A\omega_\zeta^2} \right) = 0.$$

Using the solution for ν_t the solution for consumption is

$$c_t = \frac{2A\beta R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t + \frac{\bar{c} \left(1 - A\omega_\zeta^2/\vartheta \right) + \beta B}{1 - A\omega_\zeta^2/\vartheta + 2\beta A}. \quad (78)$$

Substituting the above expressions into the Bellman equation gives

$$\begin{aligned}
& -A\widehat{s}_t^2 - B\widehat{s}_t - C \\
& = -\frac{1}{2} \left(\frac{2A\beta R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t + \frac{-2\beta A\bar{c} + \beta B}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \right)^2 \\
& + \frac{\beta\vartheta\omega_\zeta^2}{\left(2(\vartheta - A\omega_\zeta^2)\right)^2} \left[\frac{2AR(1 - A\omega_\zeta^2/\vartheta)}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t + B - \frac{2\bar{c}(1 - A\omega_\zeta^2/\vartheta)A + 2\beta AB}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \right]^2 \\
& - \beta A \left\{ \left[\frac{R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t - \frac{-B\omega_\zeta^2/\vartheta + 2c + 2B\beta}{2(1 - A\omega_\zeta^2/\vartheta + 2\beta A)} \right]^2 + \omega_\zeta^2 \right\} \\
& - \beta B \left[\frac{R}{1 - A\omega_\zeta^2/\vartheta + 2\beta A} \widehat{s}_t - \frac{-B\omega_\zeta^2/\vartheta + 2c + 2B\beta}{2(1 - A\omega_\zeta^2/\vartheta + 2\beta A)} \right] - \beta C.
\end{aligned}$$

Collecting and matching terms, the constant coefficients turn out to be

$$\begin{aligned}
A & = \frac{\beta R^2 - 1}{2\beta - \omega_\zeta^2/\vartheta}, \\
B & = \frac{(\beta R^2 - 1)\bar{c}}{(R - 1)(\omega_\zeta^2/(2\vartheta) - \beta)}, \\
C & = \frac{R(\beta R^2 - 1)}{2(\beta R - R\omega_\zeta^2/2\vartheta)(R - 1)^2} ((R - 1)\omega_\zeta^2 + \bar{c}^2).
\end{aligned}$$

When $\beta R = 1$, they reduce to

$$A = \frac{R(R - 1)}{2 - R\omega_\zeta^2/\vartheta}, \quad (79)$$

$$B = -\frac{R\bar{c}}{1 - R\omega_\zeta^2/(2\vartheta)}, \quad (80)$$

$$C = \frac{R}{2(1 - R\omega_\zeta^2/(2\vartheta))} \omega_\zeta^2 + \frac{R}{2(1 - R\omega_\zeta^2/2\vartheta)(R - 1)} \bar{c}^2. \quad (81)$$

Substituting (79) and (80) into (78) yields the consumption function (16) in the text.

The proof that $\Pi \in (0, 1)$ follows immediately from inspection of the second-order condition for the minimizing player.

7.2 Solving the Risk-Sensitive RI Model

To solve the Bellman equation (24) subject to 9, we conjecture that

$$v(\widehat{s}_t) = -C - B\widehat{s}_t - A\widehat{s}_t^2, \quad (82)$$

where A , B , and C are constants to be determined. We can then evaluate $E_t [\exp(-\alpha v(\widehat{s}_{t+1}))]$ to obtain

$$\begin{aligned}
& E_t [\exp(-\alpha v(\widehat{s}_{t+1}))] \\
&= E_t [\exp(\alpha A \widehat{s}_{t+1}^2 + \alpha B \widehat{s}_{t+1} + \alpha C)] \\
&= E_t \left[\exp \left(\alpha A (R\widehat{s}_t - c_t)^2 + \alpha B (R\widehat{s}_t - c_t) + [2\alpha A (R\widehat{s}_t - c_t) + \alpha B] \eta_{t+1} + \alpha A \eta_{t+1}^2 + \alpha C \right) \right] \\
&= (1 - 2c)^{-1/2} \exp \left(a + \frac{b^2}{2(1 - 2c)} \right),
\end{aligned}$$

where

$$\begin{aligned}
a &= \alpha A (R\widehat{s}_t - c_t)^2 + \alpha B (R\widehat{s}_t - c_t) + \alpha C, \\
b &= [2\alpha A (R\widehat{s}_t - c_t) + \alpha B] \omega_\eta, \\
c &= \alpha A \omega_\eta^2.
\end{aligned}$$

Thus, the distorted expectations operator can be written as

$$\begin{aligned}
\mathcal{R}_t [v(\widehat{s}_{t+1})] &= -\frac{1}{\alpha} \left\{ -\frac{1}{2} \log(1 - 2c) + a + \frac{b^2}{2(1 - 2c)} \right\} \\
&= \frac{1}{2\alpha} \log(1 - 2\alpha A \omega_\eta^2) - \frac{A}{1 - 2\alpha A \omega_\eta^2} (R\widehat{s}_t - c_t)^2 - \frac{B}{1 - 2\alpha A \omega_\eta^2} (R\widehat{s}_t - c_t) - \left[C + \frac{\alpha B^2 \omega_\eta^2}{2(1 - 2\alpha A \omega_\eta^2)} \right].
\end{aligned} \tag{83}$$

Maximizing the RHS of (83) with respect to c_t yields the first-order condition

$$-(c_t - \bar{c}) + \frac{2\beta A}{1 - 2\alpha A \omega_\eta^2} (R\widehat{s}_t - c_t) + \frac{B\beta}{1 - 2\alpha A \omega_\eta^2} = 0,$$

which means that

$$c_t = \frac{2A\beta R}{1 - 2\alpha A \omega_\eta^2 + 2A\beta} \widehat{s}_t + \frac{\bar{c}(1 - 2\alpha A \omega_\eta^2) + B\beta}{1 - 2\alpha A \omega_\eta^2 + 2A\beta}. \tag{84}$$

Substituting (84) and (82) into (24), and collecting and matching terms, the constant coefficients turn out to be

$$A = \frac{\beta R^2 - 1}{2\beta - 2\alpha \omega_\eta^2}, \tag{85}$$

$$B = \frac{(\beta R^2 - 1) \bar{c}}{(R - 1)(\alpha \omega_\eta^2 - \beta)}, \tag{86}$$

$$C = \frac{R(\beta R^2 - 1)}{2(\beta R - R\alpha \omega_\eta^2)(R - 1)^2} ((R - 1)\omega_\eta^2 + \bar{c}^2). \tag{87}$$

Substituting (85) and (86) into (84) yields the consumption function (25) in the text.

7.3 Solving the Robust-RI Model with Concerns about the Income Shock and the Noise

To solve the Bellman equation (19) subject to (18),²² we simply need to replace ω_ζ^2 with ω_η^2 (we omit the steps) in the constant terms obtained in Appendix 7.1:

$$\begin{aligned} A &= \frac{\beta R^2 - 1}{2\beta - \omega_\eta^2/\vartheta}, \\ B &= \frac{(\beta R^2 - 1)\bar{c}}{(R-1)(\omega_\eta^2/(2\vartheta) - \beta)}, \\ C &= \frac{R(\beta R^2 - 1)}{2(\beta R - R\omega_\eta^2/2\vartheta)(R-1)^2} \left((R-1)\omega_\eta^2 + \bar{c}^2 \right), \end{aligned}$$

which reduces to

$$\begin{aligned} A &= \frac{R(R-1)}{2 - R\omega_\eta^2/\vartheta}, \\ B &= -\frac{R\bar{c}}{1 - R\omega_\eta^2/(2\vartheta)}, \\ C &= \frac{R}{2(1 - R\omega_\eta^2/2\vartheta)}\omega_\eta^2 + \frac{R}{2(1 - R\omega_\eta^2/2\vartheta)(R-1)}\bar{c}^2. \end{aligned}$$

when $\beta R = 1$. Using (79) and (80), we can obtain the consumption function (20) in the text.

7.4 Deriving the Stochastic Properties of Consumption

7.4.1 Deriving the Volatility of Consumption

Given that

$$\Delta c_t = \frac{(1-R)\Pi}{1-\Pi}(c_{t-1} - \bar{c}) + \frac{R-1}{1-\Pi} \left[\frac{\theta\zeta_t}{1 - (1-\theta)R \cdot L} + \theta \left(\xi_t - \frac{\theta R \xi_{t-1}}{1 - (1-\theta)R \cdot L} \right) \right],$$

we can rewrite c_t as follows:

$$c_t = \rho_1 c_{t-1} + \frac{R-1}{1-\Pi} \left[\frac{\theta\zeta_t}{1 - \rho_2 \cdot L} + \theta \left(\xi_t - \frac{\theta R \xi_{t-1}}{1 - \rho_2 \cdot L} \right) \right],$$

which can be reduced to:

$$c_t = \theta \frac{R-1}{1-\Pi} \frac{\zeta_t + (\xi_t - R\xi_{t-1})}{(1 - \rho_1 \cdot L)(1 - \rho_2 \cdot L)};$$

taking unconditional variance on both sides yields

²²Note that the only difference between (15) and (18) is that the distortion term in (15) is $\omega_\zeta \nu_t$, whereas it is $\omega_\eta \nu_t$ in (18).

$$\begin{aligned}
\text{var}(c_t) &= \text{var}\left(\theta \frac{R-1}{1-\Sigma} \frac{\zeta_t + (\xi_t - R\xi_{t-1})}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \\
&= \left(\theta \frac{R-1}{1-\Sigma}\right)^2 \left[\text{var}\left(\frac{\zeta_t}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) + \text{var}\left(\frac{\xi_t - R\xi_{t-1}}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \right] \\
&= \left(\theta \frac{R-1}{1-\Sigma}\right)^2 \sum_{k=0}^{\infty} \left\{ \left[\sum_{j=0, j \leq k}^k (\rho_1^{k-j} \rho_2^j)^2 \right] \omega_{\zeta}^2 + \left[\sum_{j=0, j \leq k}^k (\rho_1^{k-j} \rho_2^j) - R \sum_{j=0, j \leq k-1}^{k-1} (\rho_1^{k-1-j} \rho_2^j) \right]^2 \omega_{\xi}^2 \right\}.
\end{aligned} \tag{88}$$

7.4.2 Deriving the Persistence of Consumption

The first-order autocovariance of consumption can be derived as follows:

$$\begin{aligned}
&\text{cov}(c_t, c_{t+1}) \\
&= \text{cov}\left(\frac{\zeta_t + (\xi_t - R\xi_{t-1})}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}, \theta \frac{R-1}{1-\Sigma} \frac{\zeta_{t+1} + (\xi_{t+1} - R\xi_t)}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \\
&= \left(\theta \frac{R-1}{1-\Sigma}\right)^2 \text{cov}\left(\frac{\zeta_t + (\xi_t - R\xi_{t-1})}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}, \frac{\zeta_{t+1} + (\xi_{t+1} - R\xi_t)}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \\
&= \left(\theta \frac{R-1}{1-\Sigma}\right)^2 \text{cov}\left(\frac{\zeta_t}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)} + \frac{\xi_t - R\xi_{t-1}}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}, \frac{\zeta_{t+1}}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)} + \frac{\xi_{t+1} - R\xi_t}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \\
&= \left(\theta \frac{R-1}{1-\Sigma}\right)^2 \left[\text{cov}\left(\frac{\zeta_t}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}, \frac{\zeta_{t+1}}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \right. \\
&\quad \left. + \text{cov}\left(\frac{\xi_t - R\xi_{t-1}}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}, \frac{\xi_{t+1} - R\xi_t}{(1-\rho_1 \cdot L)(1-\rho_2 \cdot L)}\right) \right] \\
&= \left(\theta \frac{R-1}{1-\Sigma}\right)^2 \sum_{k=0}^{\infty} \left(+ \left\{ \begin{array}{l} \left[\sum_{j=0, j \leq k}^k (\rho_1^{k-j} \rho_2^j) \right] \left[\sum_{j=0, j \leq k+1}^{k+1} (\rho_1^{k+1-j} \rho_2^j) \right] \omega_{\zeta}^2 \\ \left[\sum_{j=0, j \leq k}^k (\rho_1^{k-j} \rho_2^j) - R \sum_{j=0, j \leq k-1}^{k-1} (\rho_1^{k-1-j} \rho_2^j) \right] \cdot \\ \left[\sum_{j=0, j \leq k+1}^{k+1} (\rho_1^{k+1-j} \rho_2^j) - R \sum_{j=0, j \leq k}^k (\rho_1^{k-j} \rho_2^j) \right] \end{array} \right\} \omega_{\xi}^2 \right)
\end{aligned}$$

where we use the facts that

$$\begin{aligned}
& \text{cov} \left(\frac{\zeta_t}{(1 - \rho_1 \cdot L)(1 - \rho_2 \cdot L)}, \frac{\zeta_{t+1}}{(1 - \rho_1 \cdot L)(1 - \rho_2 \cdot L)} \right) \\
&= \text{cov} \left(\sum_{k=0}^{\infty} \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \zeta_{t-k}, \sum_{k=0}^{\infty} \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \zeta_{t+1-k} \right) \\
&= \text{cov} \left(\sum_{k=0}^{\infty} \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \zeta_{t-k}, \sum_{k=0}^{\infty} \left[\sum_{j=0, j \leq k+1}^{k+1} \left(\rho_1^{k+1-j} \rho_2^j \right) \right] \zeta_{t-k} \right) \\
&= \sum_{k=0}^{\infty} \left\{ \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \left[\sum_{j=0, j \leq k+1}^{k+1} \left(\rho_1^{k+1-j} \rho_2^j \right) \right] \right\} \omega_{\zeta}^2.
\end{aligned}$$

and

$$\begin{aligned}
& \text{cov} \left(\frac{\xi_t - R\xi_{t-1}}{(1 - \rho_1 \cdot L)(1 - \rho_2 \cdot L)}, \frac{\xi_{t+1} - R\xi_t}{(1 - \rho_1 \cdot L)(1 - \rho_2 \cdot L)} \right) \\
&= \text{cov} \left(\begin{aligned} & \xi_t + \sum_{k=1}^{\infty} \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) - R \sum_{j=0, j \leq k-1}^{k-1} \left(\rho_1^{k-1-j} \rho_2^j \right) \right] \xi_{t-k}, \\ & \xi_{t+1} + \sum_{k=1}^{\infty} \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) - R \sum_{j=0, j \leq k-1}^{k-1} \left(\rho_1^{k-1-j} \rho_2^j \right) \right] \xi_{t+1-k} \end{aligned} \right) \\
&= \sum_{k=0}^{\infty} \left\{ \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) - R \sum_{j=0, j \leq k-1}^{k-1} \left(\rho_1^{k-1-j} \rho_2^j \right) \right] \left[\sum_{j=0, j \leq k+1}^{k+1} \left(\rho_1^{k+1-j} \rho_2^j \right) - R \sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \right\} \omega_{\xi}^2.
\end{aligned}$$

where $\sum_{j=0, j \leq k-1}^{k-1} \left(\rho_1^{k-1-j} \rho_2^j \right) = 0$ if $k = 0$. We can then compute the first-order autocorrelation of consumption as follows:

$$\rho(c_t, c_{t+1}) = \frac{\sum_{k=0}^{\infty} \left(\begin{aligned} & \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \left[\sum_{j=0, j \leq k+1}^{k+1} \left(\rho_1^{k+1-j} \rho_2^j \right) \right] \\ & + \left\{ \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) - R \sum_{j=0, j \leq k-1}^{k-1} \left(\rho_1^{k-1-j} \rho_2^j \right) \right] \cdot \right. \\ & \left. \left[\sum_{j=0, j \leq k+1}^{k+1} \left(\rho_1^{k+1-j} \rho_2^j \right) - R \sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right] \right\} \left(\omega_{\xi}^2 / \omega_{\zeta}^2 \right) \end{aligned} \right)}{\sum_{k=0}^{\infty} \left\{ \begin{aligned} & \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) \right]^2 \\ & + \left[\sum_{j=0, j \leq k}^k \left(\rho_1^{k-j} \rho_2^j \right) - R \sum_{j=0, j \leq k-1}^{k-1} \left(\rho_1^{k-1-j} \rho_2^j \right) \right]^2 \left(\omega_{\xi}^2 / \omega_{\zeta}^2 \right) \end{aligned} \right\}}.$$

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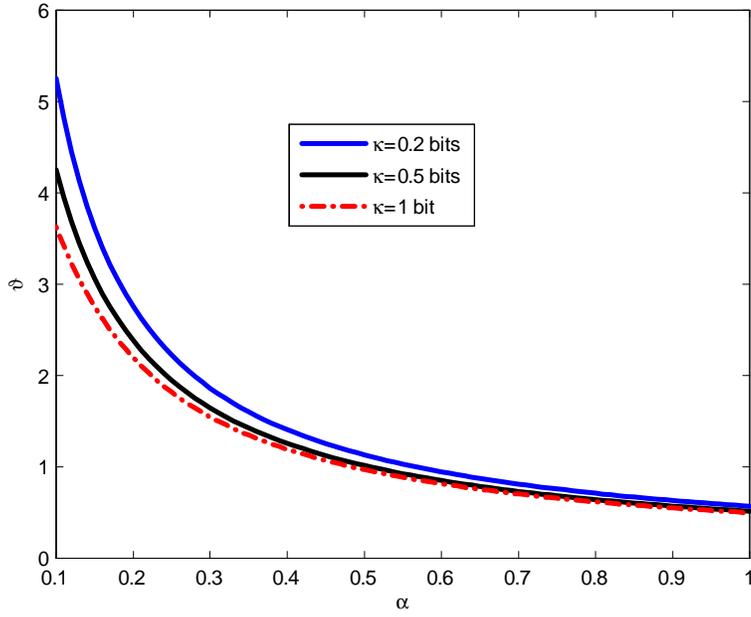


Figure 1: Nonlinear OE between RB and RS

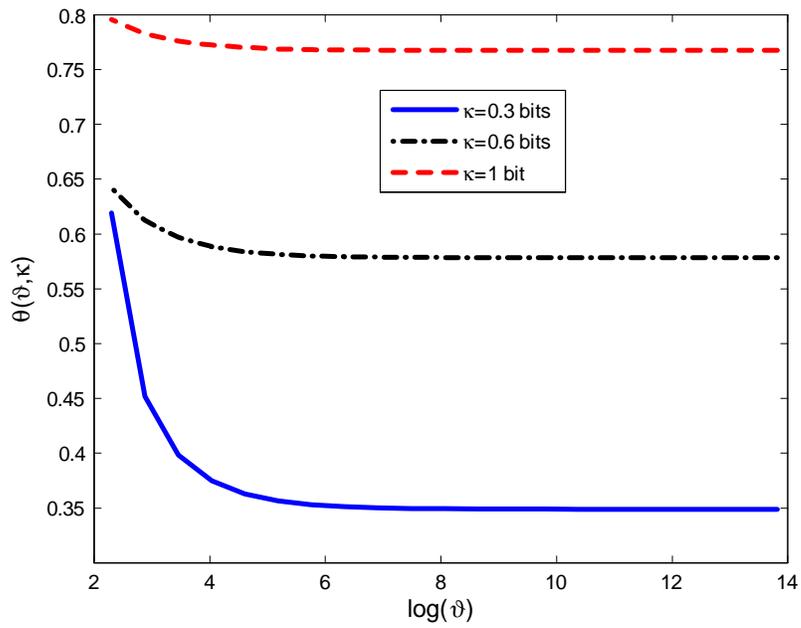


Figure 2: Effects of RB and RI on Robust Kalman Gain θ

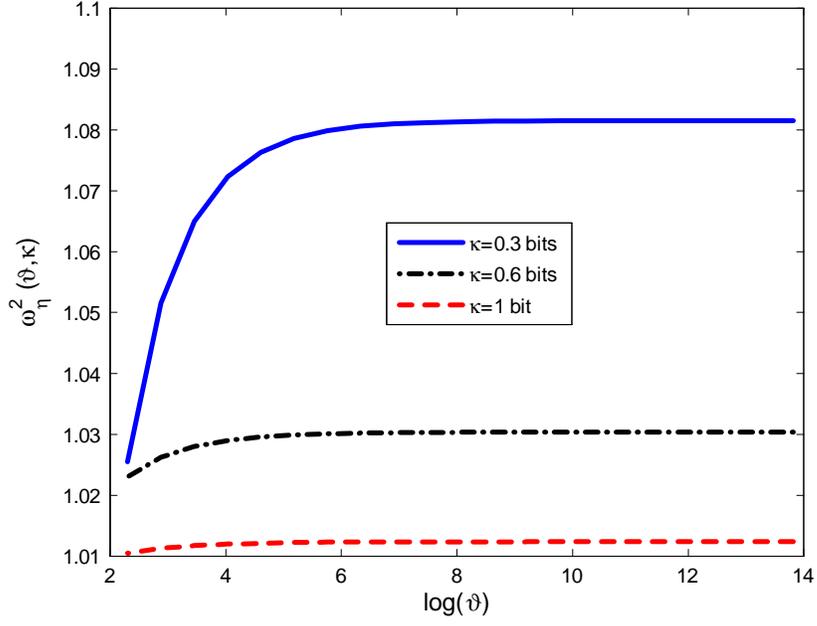


Figure 3: Effects of Robustness and Capacity, ϑ and κ , on ω_η^2

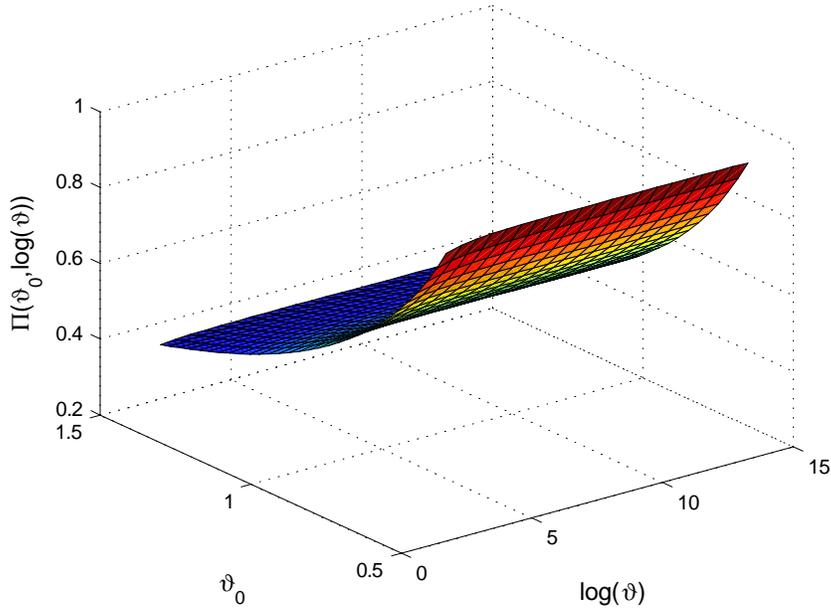


Figure 4: Effects of Two Types of Robustness, ϑ_0 and ϑ , on Π

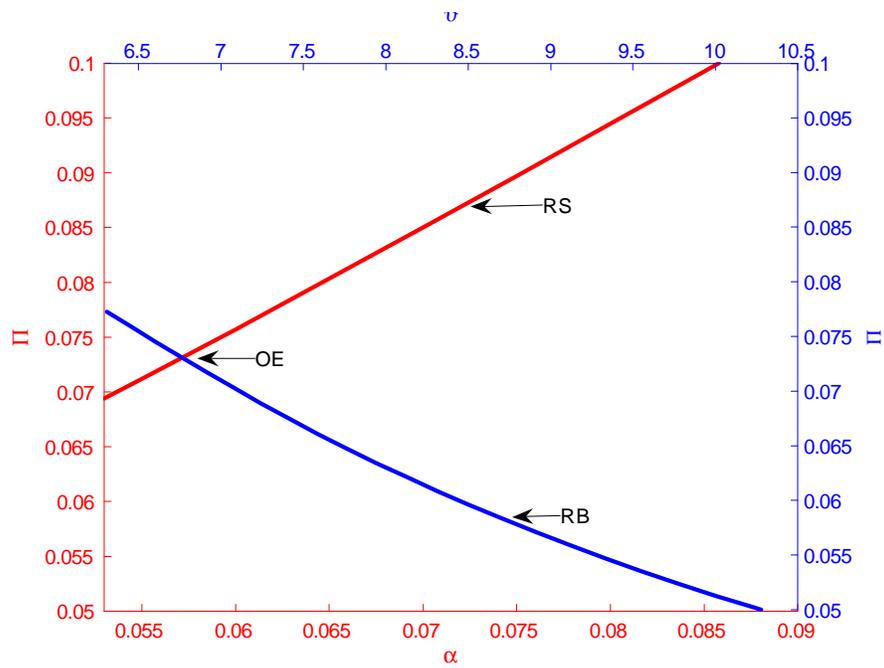


Figure 5: Nonlinear OE between RB and RS

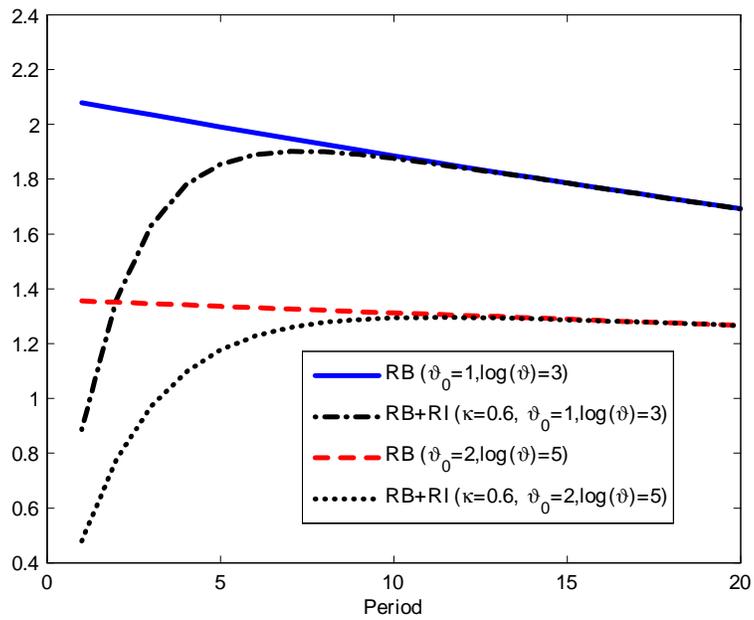


Figure 6: Impulse Responses of Consumption to Income Shock

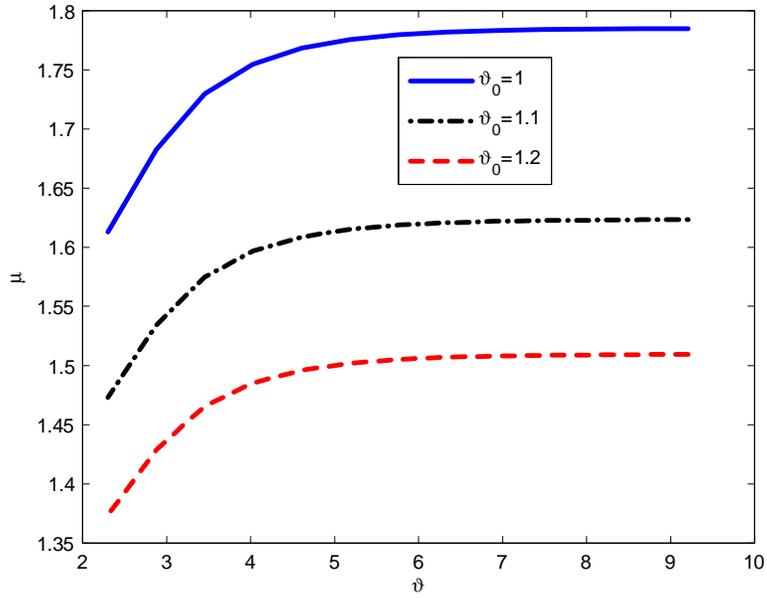


Figure 7: Relative Volatility of Consumption to Income under RB-RI

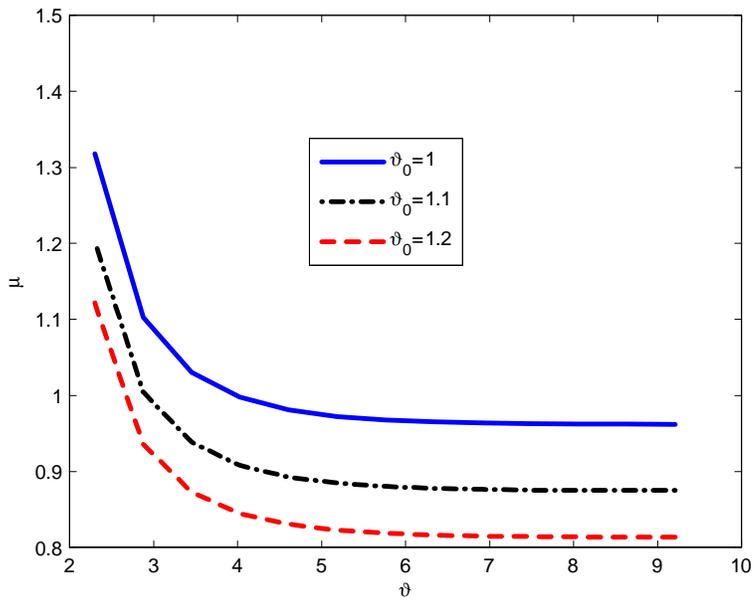


Figure 8: Relative Volatility of Consumption to Income under RB-RI without Noise

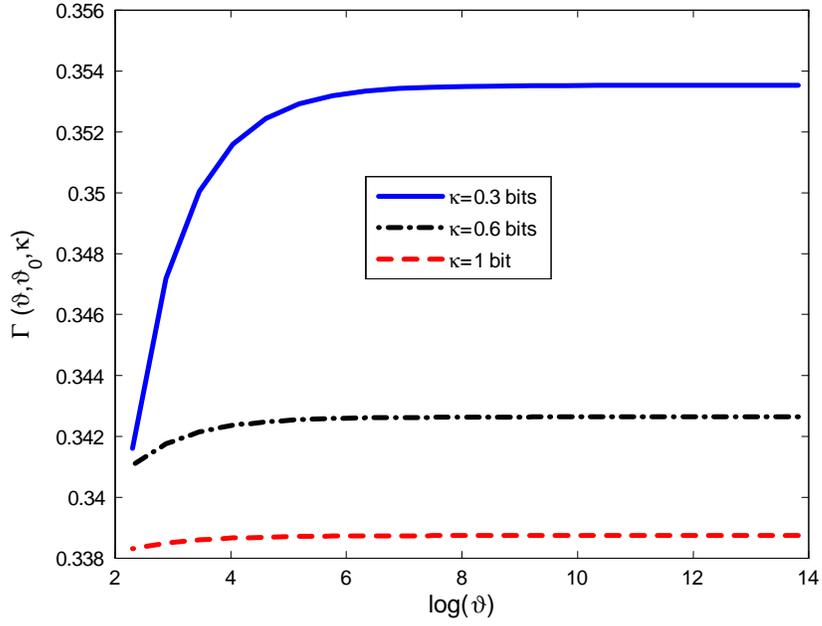


Figure 9: Effects of ϑ and κ on Welfare Costs

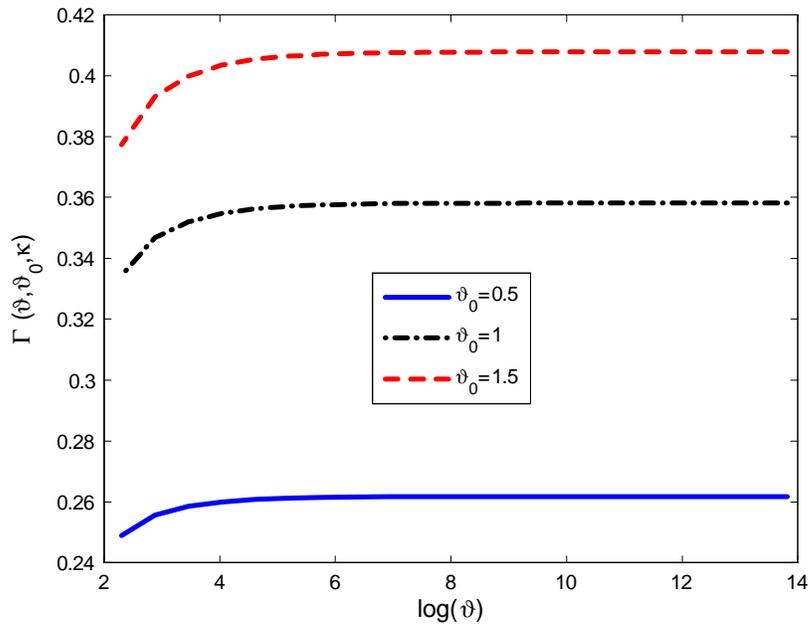


Figure 10: Effects of ϑ_0 , ϑ , and κ on Welfare Costs

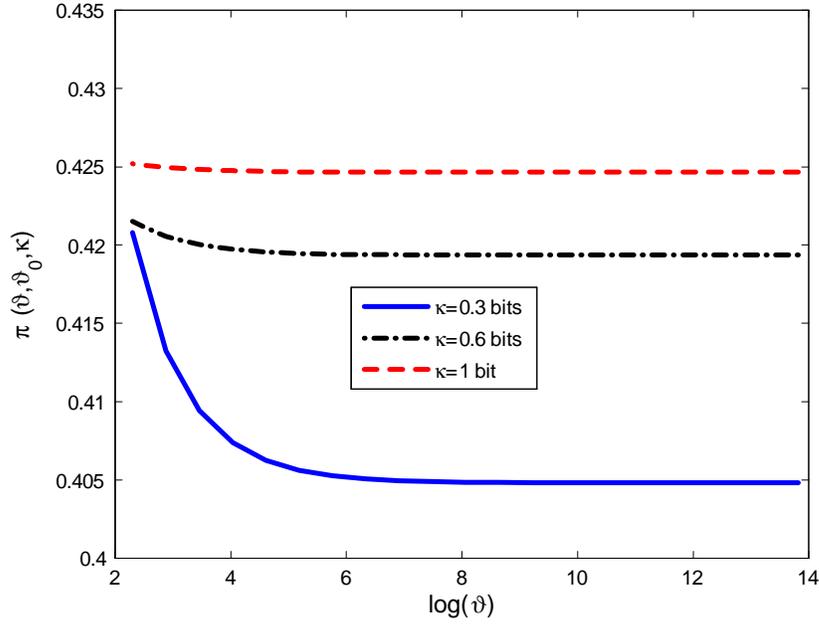


Figure 11: Effects of ϑ and κ on Relative Marginal Welfare Losses

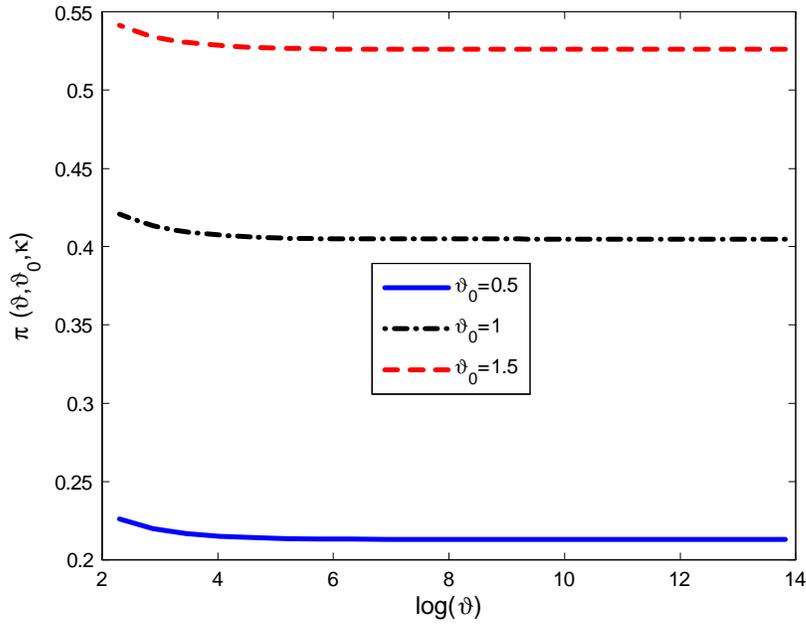


Figure 12: Effects of ϑ and ϑ_0 on Relative Marginal Welfare Losses