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Belief updating under ambiguity: A numerical simulation analysis

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ABSTRACT

We compare Bayesian with the two non-Bayesian updating under ambiguity in its model specification and numerical implications. For an ambiguity averse (seeking) decision maker, we find that Gilboa and Schmeidler (GS) updating leads to a lower (higher) certainty equivalent (CE) than Dempster–Shafer (DS) updating and Bayesian updating, and the change of historical information has a bigger (smaller) impact on GS updating than on DS updating. In addition, the results shows that GS updating is more sensitive to the ambiguity attitude than DS updating. The application of numerical analysis in finance especially implies that the sensitivity of price to information depends on the ambiguity attitude of the market.

1. Introduction

The traditional Bayesian updating approach (abbreviated as BS) assumes ambiguity neutral behavior, and provides a simple method to calculate posterior probabilities based on prior beliefs and received information. Neither ambiguity averse nor ambiguity seeking behavior can be accommodated. Much literature (Trautmann and Van De Kuilen, 2015; Baillon et al., 2018) has shown that a large proportion of people are ambiguity averse, with also considerable ambiguity seeking, showing the need to develop non-Bayesian approaches. Dempster (1967) and Shafer (1976) (abbreviated as DS) provided a famous non-Bayesian updating rule. They used non-additive measures of belief and weighting functions to represent perceived likelihood. Another famous non-Bayesian updating rule was provided by Gilboa and Schmeidler (1993) (abbreviated as GS). Different markets and people may have different belief updating rules about new information, so it is necessary to compare and analyze the difference among the three aforementioned belief updating rules widely studied in decision theory.

This paper use the framework of decision theory to compare the three belief updating rules (Bayesian and non-Bayesian updating) in its model specification and numerical implications. It shows that the DM would be more willing to pay for the prospect in the future if the outcomes increase or the probabilities of good historical information[1] increase, which is along with the intuition. Furthermore, GS updating is more sensitive to the change of ambiguity attitude, i.e., GS updating accentuates deviations from ambiguity neutrality relative to DS updating and Bayesian updating. And the main conclusion of this paper is that under ambiguity aversion (seeking), BS updating leads to the highest (lowest) CE, while GS updating leads to a lowest (highest) CE and are more impacted by the change of the information from the worst history to the best history.

We focus on the numerical simulation analysis of the three belief updating rules for three reasons. First, it is difficult to theoretically prove their ranking of belief updating rules under ambiguity due to the complexity of the problem. A rich literature has focused on belief updating rules, but little has theoretically proven the ranking of various belief updating rules. Second, in reality, there is a lack of experimental or empirical data to directly test the ranking of belief updating rules. It is hard to test the ranking of belief updating rules under ambiguity through experiments since it is difficult to elicit ambiguous beliefs and identify which exact belief updating rule a DM follows, and further observe how they are affected by historical information based on existing studies. Finally, numerical simulation has some advantages. It can accomplish tasks that cannot be accomplished through experiments, and the cost is relatively low. Most importantly, numerical simulation can use specific numerical values in a more general settings to more intuitively display

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[1] In this paper, we called good historical information if event corresponding to good outcome occurred in one period or in two period simultaneously, and bad historical information if event corresponding to bad outcome occurred in one period or in two period simultaneously. For example, conditional A (event A happened on day 0 or day 1) means good historical information if the outcome of event A is high, while conditional C (event C happened on day 0 or day 1) means bad historical information if the outcome of event C is low.

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conclusions that are not easily discovered, making it easier for people to understand and analyze.

This paper is organized as follows. Section 2 presents the models of probabilities and the belief updating. A numerical analysis of the three updating approaches is in Section 3. Section 4 concludes.

2. The models of probabilities and belief updating

2.1. Probability measures

We consider three exhaustive and mutually exclusive events A, B and C and use symbols α, β, γ to denote general elements of $\{A, B, C\}$. Further, α_0 denotes the generic event α realized on day 0, β_1 the generic event β realized on day 1, γ_2 the generic event γ realized on day 2. In each day, one of the generic events is realized, so that a state (of nature) $s \in S$ can be expressed by $(\alpha_0, \beta_1, \gamma_2)$[2].

The calculation of the prior probability belong to the Dirichlet family[3], which is as follows:

$$P(\alpha_0, \beta_1, \gamma_2) = P(\gamma_2|\alpha_0, \beta_1) \cdot P(\beta_1|\alpha_0) \cdot P(\alpha_0). \tag{1}$$

where $\alpha, \beta, \gamma \in \{A, B, C\}$, and the subscript refers to the day. The conditional probabilities are that the DM predicts future events based on her prior belief and the observation of new events, and are calculated as follows[4]:

$$P(\gamma_2|\alpha_0, \beta_1) = \frac{\lambda P(\gamma_0) + n_2}{\lambda + 2} \text{and} P(\beta_1|\alpha_0) = \frac{\lambda P(\beta_0) + n_1}{\lambda + 1} \tag{2}$$

where λ is the parameter that represents the weight on one's prior belief[5] and we here assume $\lambda = 2$; n_1 is the number of appearances of generic event α in the past 1 day (day 0); n_2 is the number of appearances of generic event β in the past 2 days (day 0 and day 1). Probabilities of two states of nature are the same if the total number of each generic event occurring is the same, e.g., $P(A_0, C_1, C_2) = P(C_0, C_1, A_2)$.

2.2. Updating rules

2.2.1. Bayesian updating approach

For each updating method, we first discuss a general conditional weighting function W_c, and then turn to the special case of CS-RDU considered in this paper. All conditional weighting functions that we will consider are the following general form defined by Sarin and Wakker (1998):

$$W_c(E|G) = \frac{\pi(E \cap G)}{\pi(G)}. \tag{3}$$

Here we have $E, G \subset S$, the π is decision weights, and $W_c() \in [0, 1]$. The decision weights π differ for the different methods, and will be explained for each case. The differences can be interpreted as different assumptions about the ranks of the events. And the Bayesian updating could be expressed as:

$$W_b(E|G) = P(E|G). \tag{4}$$

2.2.2. Gilboa and Schmeidler's updating approach

Gilboa (1989a, b) proposed the following rule to update belief:

$$W_g(E|G) = \frac{W(E \cap G)}{W(G)}, \tag{5}$$

where $E, G \subset S$, W is weighting function and $W_g(|G)$ denotes the updated weighting function conditional on event G. This updating rule was shown to be plausible and was axiomatized by Gilboa and Schmeidler (1993). The underlying assumption in this rule is that the DM assumes that the event G, of which she has been informed, corresponds with the "best of all possible outcomes". The corresponding rank-ordering of events is $E \bigcap G \succcurlyeq G \backslash E \succcurlyeq G^c$. By applying this rank order in Eq. (3), we obtain

$$W_c(E|G) = \frac{\pi(E \cap G)}{\pi(G)} = \frac{W(E \cap G)}{W(G)} = W_g(E|G). \tag{6}$$

This shows that the general conditional weighting function coincides with GS updating if a specific ranking of events is assumed.

If we assume CS-RDU with the source function a power function, i.e. $W(E) = w(P(E)) = P(E)^r$, then G&S updating can be expressed by

$$\begin{aligned} W_g(E|G) &= \frac{W(E \cap G)}{W(G)} = \frac{P(E \cap G)^r}{P(G)^r} \\ &= \frac{(P(E|G)P(G))^r}{P(G)^r} = P(E|G)^r. \end{aligned} \tag{7}$$

For GS updating, an increase in r results directly in a decrease in $W_g(E|G)$. A small change in r has considerable impact on $W_g(E|G)$, which the numerical analysis reported later will confirm. As appears from Eq. (7), the conditional weight is fully determined by the probabilities of the states of nature.

2.2.3. Dempster and shafer's updating approach

Dempster (1967) and Shafer (1976) proposed another rule to update belief:

$$W_d(E|G) = \frac{W((E \cap G) \cup G^c) - W(G^c)}{1 - W(G^c)} \tag{8}$$

where E, $G \subset S$, W is the weighting function an $W_d(\cdot|G)$ denotes the updated weighting function conditional on G. This rule is referred to as *Dempster–Shafer (DS)* updating. The assumption underlying this rule is that the DM assumes that the event G, of which she has been informed, corresponds with the "worst of all possible outcomes". The corresponding rank-ordering of events is $G^c \succcurlyeq E \bigcap G \succcurlyeq G \backslash E$. Then

$$W_c(E|G) = \frac{\pi(E \cap G)}{\pi(G)} = \frac{W((E \cap G) \cup G^c) - W(G^c)}{1 - W(G^c)} = W_d(E|G). \tag{9}$$

This shows that the general conditional weighting function coincides with DS updating when a specific ranking of events is assumed. Similarly, the DS approach can be expressed by

$$\begin{aligned} W_d(E|G) &= \frac{W((E \cap G) \cup G^c) - W(G^c)}{1 - W(G^c)} \\ &= \frac{(P(E \cap G) + P(G^c))^r - P(G^c)^r}{1 - P(G^c)^r}. \end{aligned} \tag{10}$$

Unlike GS updating, the final result cannot be reduced to a (power) function of a conditional probability. An increase in r results in a decrease in $P(G^c)$, so that the denominator increases. However, the effect of power r on the conditional weighting function $W_d(E|G)$ is not directly clear. Therefore, $W_d(E|G) = W_d(E|F)$ if the total number of each generic event occurring in G and E is the same. In particular, again, the order of the observed generic events does not affect conditional weight.

3. Numerical simulation analysis

This section uses numerical simulation methods with more general settings of outcomes, probabilities and the power r to compare the three

[2] So $P(\alpha_0)$ denotes the probability of the realization of generic event α on day 0. Similarly, $P(\alpha_0, \beta_1)$ is the probability of the realization of generic event α on day 0 and β on day 1, and $P(\alpha_0, \beta_1, \gamma_2)$ means the probability of the realization of generic event α on day 0, β on day 1, and γ on day 2.

[3] It is a multinomial extension of the beta family, a widely used conjugate family (Wilks, 1962).

[4] The methods of calculation are special cases of Carnap's induction (1952, 1980).

[5] That is, the level of trust the decision maker gives to her prior belief.

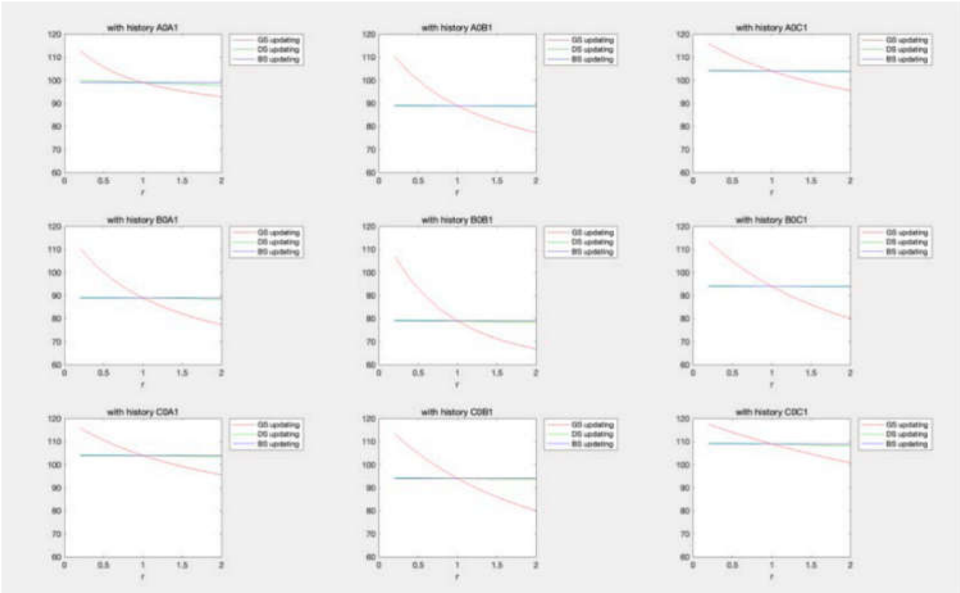

Fig. 1. The CEs of the three updating rules on day 2 with different powers.

updating rules. We still consider the framework described above with a normalized outcome (100, O_B, O_C) corresponding to generic events A, B, and C, and probabilities on day 0 with $P(B_0) = p_B$, $P(C_0) = p_C$, $P(A_0) = p_A = 1 - p_B - p_C$. We now generalize the approach for any $O_B, O_C \in [0, 200]$ where $O_A = 100$; $p_B, p_C \in [0, 1]$ where $p_B + p_C \leq 1$; $r \in [0, 2]$ where $r > 1$ concerns convex weighting and ambiguity aversion, $r < 1$ concerns concave weighting and ambiguity seeking. It is intuitively concluded that all CEs increase with the increasing of each outcome and the increasing of the probability of the event with good outcome (while with the decreasing of the probability of the event with bad outcome).[6]

3.1. *The influence of the power r*

In this section, we assume that $p_A = 0.5, p_B = 0.2, p_C = 0.3$ and $O_A = 100, O_B = 60, O_C = 120$ without losing generality[7]. As the impact of r on DS updating is not clear from the model specification, Fig. 1 is confirmed by the numerical results: GS updating is more sensitive to the change of r and, hence, the change of r has only a larger effect on GS than DS updating. Moreover, the CE of GS updating is higher when ambiguity seeking ($r < 1$) and lower when ambiguity averse ($r > 1$).[8] This confirms that a small change in r has a considerable impact on CEs. The change in DS updating is less significant when r changes and the CE of DS updating is slightly lower (higher) than Bayesian updating under ambiguity averse (seeking). Thus, GS updating accentuates deviations from ambiguity neutrality relative to DS updating.

When we compare different cases on day 2, we find that whether good or bad historical information affects CEs of prospects differently between GS and DS updating. Compared with the case when good historical information is received,[9] CEs of the three updating rules are higher than bad historical information.[10] Besides, under ambiguity averse ($r > 1$), a DM's CE of the prospect increases more under GS updating than under DS updating when good historical information is received.[11] It can be found that compared the green line (the CEs of the DS updating) and blue line (the CEs of the Bayesian updating), the red line denoting GS updating increase less when $r < 1$ and increase more when $r > 1$. And the same still holds true for the cases of information changing from B_0 to C_0 on day 0 or from B_1 to C_1 on day 1. This implies that the change of historical information has a bigger impact on GS updating than on DS updating for an ambiguity averse DM and has a smaller impact on GS updating than on DS updating for an ambiguity seeking DM.

3.2. *The comparison of the CEs*

And we discuss how 5 parameters change simultaneously influences the results.[12] In Fig. 2,[13] Rank 1 denotes the ranking: $CE_{GS \quad updating} \geq CE_{DS \quad updating} \geq CE_{Bayesian \quad updating}$, and Rank 2 denotes the ranking: $CE_{Bayesian \quad updating} \geq CE_{DS \quad updating} \geq CE_{GS \quad updating}$. We could conclude that the CEs in the three updating on day 2 are always follow Rank 1 under ambiguity seeking ($r < 1$) while are follow Rank 2 when ambiguity aversion ($r > 1$) with the history (A_0, A_1). The conclusion could be extended to other histories and it is consistent with the results we analyze above. Thus under ambiguity aversion (seeking), BS updating leads to the highest (lowest) CE, while GS updating leads to a lowest (highest) CE.

4. Conclusions

This paper discusses belief updating and its implications in financial pricing under different approaches. The Bayesian updating rule only

[6] The simulation results are given in appendix.

[7] We only show the results with $p_A = 0.5, p_B = 0.2, p_C = 0.3$ and $O_A = 100$, $O_B = 60, O_C = 120$ because of the page limitation. The findings still hold up with other probabilities and outcomes.

[8] $r = 1$ concerns linear weighting and ambiguity neutral which is Bayesian updating.

[9] The historical event with good outcome occurred, i.e., the event C_0 and C_1.

[10] The historical event with bad outcome occurred, i.e., the event B_0 and B_1.

[11] To better illustrate the finding, we could see the change from 5th subfigure with history (B_0, B_1) to 9th subfigure with history (C_0, C_1).

[12] As the CEs are the 6-dimensional data, it is difficult to show all data in graphical form (see more details in Online Supplement)

[13] Fig. 2 only shows the rank of the CEs with the changing of all possible probabilities and outcomes with the history (A_0, A_1). Due to space limitation, we here only show very finite discrete changings and there are more results on Online Supplement. And the number of the points decrease when p_C increase because of the properties of probability $p_B + p_C \leq 1$.

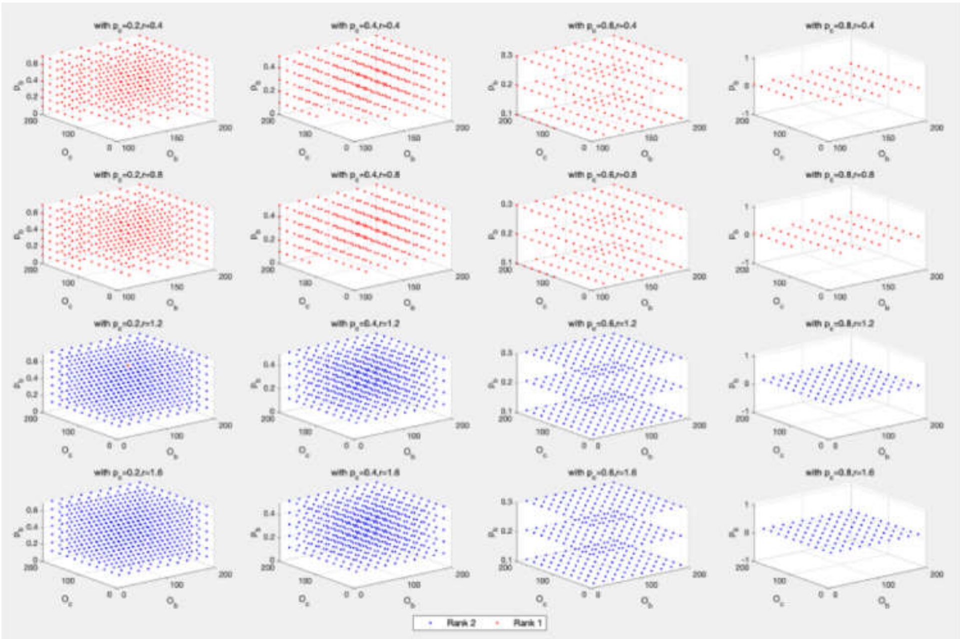

Fig. 2. The ranking of the CEs with the history (A_0, A_1).

allows for ambiguity neutral behavior while the GS and DS approaches are more flexible and can accommodate both ambiguity averse and ambiguity seeking behavior. The Bayesian approach is the special case of GS and DS approaches when ambiguity neutrality holds. A numerical analysis is provided to compare the three updating rules. Ambiguity attitudes affect not only static decisions, but also the way in which new information is incorporated. For an ambiguity averse (seeking) DM, GS updating leads to a lower (higher) CE than DS updating, and information changing has a bigger (smaller) impact on GS updating than on DS updating.

Our findings can be applied in other domains, especially finance. Then we take the market as the DM and prospects concern state-contingent assets. The CE of a prospect then is its market price, which depends on the market's beliefs. For an ambiguity seeking market, the price of the prospect is higher under GS updating than under DS updating. If the market is ambiguity averse, then the price of the prospect is lower under GS updating than under DS updating. That is, compared to agents with DS updating, agents with GS updating are more pessimistic in an ambiguity averse market and more optimistic in an ambiguity seeking market. As Bayesians are always neutral and do not overestimate or underestimate prices, they will be remain as survivors while DS/GS agents are eventually driven from the market in the long term.

Prices under DS updating are relatively stable no matter what the market's ambiguity attitude is, and they are always close to Bayesian updating. If we assume Bayesian updating to be a benchmark for rationality, then GS updating leads to more deviations from rationality than DS updating, and it is more sensitive to the ambiguity attitude of the market. A small change in ambiguity attitude of the market leads to a large change in price under GS updating no matter what its ambiguity attitude is. And the sensitivity of price to information depends on the ambiguity attitude of the market under GS updating. For an ambiguity seeking market, good (bad) historical information leads to the smallest rise (decrease) in price under GS updating than other two updating rules. For an ambiguity averse market, good (bad) historical information leads to the biggest rise (decrease) in price under GS updating than other two updating rules.

In conclusion, this paper uses numerical methods to compare the differences among three classic belief updating (Bayesian and non-Bayesian updating) methods under ambiguity, which may help to understand why individuals exhibit different behaviors after obtaining new information under ambiguity. For the convenience of comparing differences between Bayesian and non-Bayesian updating, the methods we have selected like DS/GS may be considered as simple and then have some limitations compared to the other common approaches in practice for updating such as the 'recursive multiple priors' approach of Epstein and Schneider (2007) and the 'robust filtering' approach of Hansen and Sargent (2001). First, GS/DS updating leave no role of confidence on the inference of information, the ES approach allows the confidence of learning over time and HS approach allows the confidence of model selection. Thus, compared to the agents with GS updating, the agents with ES/HS updating are less pessimistic (i.e., the price of prospect are lower under GS updating than ES/HS updating in an ambiguity averse market) due to introducing of confidence. Second, ES/HS rules are dynamic consistent in somewhat different senses which can immune to 'Dutch Book' exploitation, whereas DS/GS rules do not possess this immunity. In the future, it is essential to compare relatively richer updating rules under ambiguity.

Data availability

Data will be made available on request.

Acknowledgments

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.econlet.2023.111359.

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